THE CUMULATIVE DISTRIBUTION FUNCTION OF THE LEFT-TRUNCATED NORMAL DISTRIBUTION

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Definition of the Standardized, Left-Truncated Normal Distribution

Consider a normally-distributed random variable x with a probability density function f(x) specified as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty \quad (1)$$

If the values of x below some value x_L cannot be observed – due to censoring or truncation – then, as shown in Figure 1, the resulting distribution is a left-truncated normal distribution with probability density function $f_{LTN}(x)$ given by

$$f_{LTN}(x) = \begin{cases} 0, & -\infty \le x \le x_L \\ \dfrac{f(x)}{\int_{x_L}^{\infty} f(x)dx}, & x_L \le x \le \infty \end{cases} \quad (2)$$

where f(x) is as defined in Equation 1.

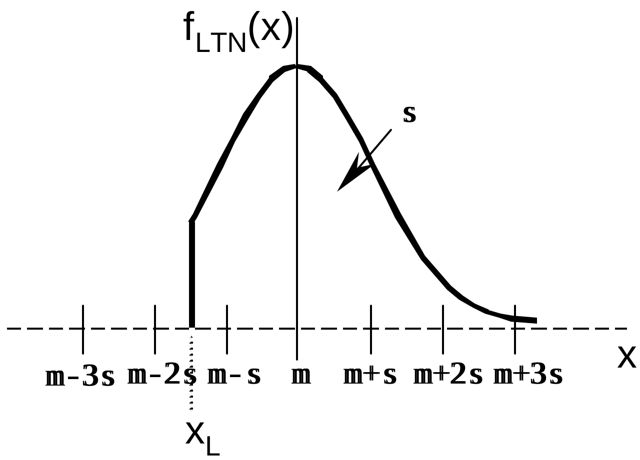

Figure 1. Left-Truncated Normal Distribution (in Terms of the Original Population Parameters)

For purposes of generality, Equation 2 can be re-stated in terms of the standard normal distribution (denoted f(z)) where

$$z = \frac{x - \mu}{\sigma} \tag{3}$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty \leq z \leq \infty \tag{4}$$

In terms of this standard normal distribution, the point of truncation x_L will be denoted k_L as given by

$$k_L = \frac{x_L - \mu}{\sigma} \tag{5}$$

Reformulating the left-truncated normal distribution of Equation 2 in terms of the standard normal distribution, the following can be found:

$$f_{LTN}(z) = \begin{cases} 0, & -\infty \le z \le k_L \\ \dfrac{f(z)}{\int_{k_L}^{\infty} f(z)dz}, & k_L \le z \le \infty \end{cases} \qquad (6)$$

This is illustrated in Figure 2.

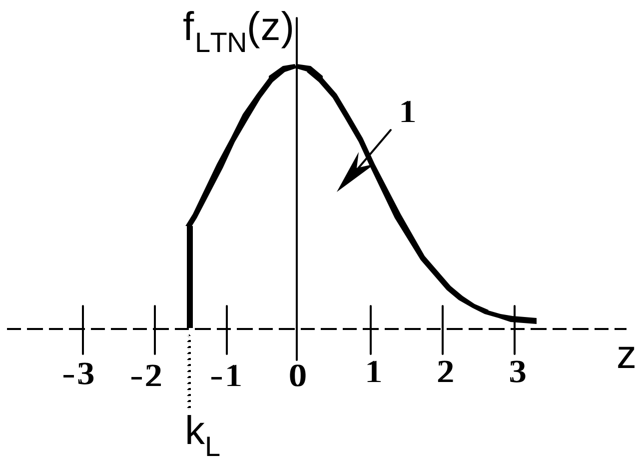

Figure 2. Left-Truncated Normal Distribution (in Terms of the Standard Normal Distribution)

To define what this paper terms as a “standardized, left-truncated normal distribution,” a standardizing variable $t = z - k_L$ is introduced, which has the effect of defining the point of truncation as $t = 0$ as in [Tho80]. The standardized, left-truncated normal distribution $f_{SLTN}(t)$ is, thus, given by

$$f_{SLTN}(t) = \begin{cases} 0, & t \leq 0 \\ \dfrac{f(t + k_L)}{\int_{k_L}^{\infty} f(z)dz}, & t \geq 0 \end{cases} \qquad (7)$$

and is illustrated in Figure 3.

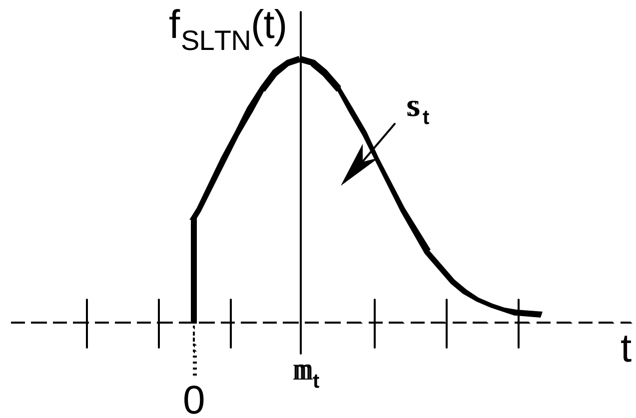

Figure 3. Standardized, Left-Truncated Normal Distribution

This paper, next, develops formulas for the two parameters of the standardized, left-truncated normal distribution: the mean (μ_t) and the standard deviation (σ_t) – for a given point of truncation k_L.

Parameters of the Standardized, Left-Truncated Normal Distribution

Consider the mean (μ_t) of the standardized, left-truncated normal distribution – for a given point of truncation k_L, where f(t) is as defined in Equation 7.

$$\mu_t = E(t) = \int_0^{\infty} t f(t) dt \tag{8}$$

With $t = z - k_L$, it follows that $dt = dz$, $z = k_L$ for $t = 0$, and $z = \infty$ for $t = \infty$. Using this, Equation 8 becomes

$$\begin{aligned} \mu_t &= \frac{1}{\int_{k_L}^{\infty} f(z)dz} \int_{k_L}^{\infty} (z - k_L) f(z) dz \\ &= \frac{1}{\int_{k_L}^{\infty} f(z)dz} \left[\int_{k_L}^{\infty} z f(z) dz - k_L \int_{k_L}^{\infty} f(z) dz \right] \end{aligned} \tag{9}$$

Consider the first term within the brackets in Equation 9. To evaluate this term, one can make use of the identity [Bey87]

$$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \tag{10}$$

Using Equations 4 and 10, it can be shown that

$$\int_{k_L}^{\infty} z f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{k_L}^{\infty} z e^{-\frac{1}{2}z^2} dz = \sqrt{\frac{2}{\pi}} \int_{k_L/\sqrt{2}}^{\infty} x e^{-x^2} dx$$
$$= -\frac{1}{\sqrt{2\pi}} e^{-x^2} \Big|_{k_L/\sqrt{2}}^{\infty} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k_L^2} = f(k_L) \quad (11)$$

Defining H(k) as

$$H(k) = \int_{k}^{\infty} f(z) dz \quad (12)$$

and using Equations 11 and 12, Equation 9 can be re-stated in its final form as

$$\mu_t = \frac{1}{H(k_L)} \left[f(k_L) - k_L H(k_L) \right] \quad (13)$$

From Equation 13, it is clear that the mean of the standardized, left-truncated normal distribution is uniquely determined by and solely dependent upon the point of truncation k_L.

Consider now the standard deviation (σ_t) of the standardized, left-truncated normal distribution. Given that [Hin90]

$$\sigma_t^2 = E(t^2) - \mu_t^2 \quad (14)$$

It is only necessary to find $E(t^2)$.

$$E(t^2) = \int_0^\infty t^2 f(t)dt = \frac{1}{H(k_L)} \int_{k_L}^\infty (z - k_L)^2 f(z)dz$$
$$= \frac{1}{H(k_L)} \left[\int_{k_L}^\infty z^2 f(z)dz - 2k_L \int_{k_L}^\infty z f(z)dz + k_L^2 \int_{k_L}^\infty f(z)dz \right] \quad (15)$$

The last two terms within the brackets in Equation 15 have already been encountered in Equations 11 and 12, respectively. Consider, now, the first term – which, via integration by parts, can be found to be equivalent to

$$\int_{k_L}^\infty z^2 f(z)dz = -z f(z)\Big|_{k_L}^\infty + \int_{k_L}^\infty f(z)dz = k_L f(k_L) + H(k_L) \quad (16)$$

Utilizing Equations 11, 12, and 16, Equation 15 can be re-stated as

$$E(t^2) = \frac{1}{H(k_L)} \left[k_L f(k_L) + H(k_L) - 2k_L f(k_L) + k_L^2 H(k_L) \right]$$
$$= \frac{1}{H(k_L)} \left[(1 + k_L^2) H(k_L) - k_L f(k_L) \right] \quad (17)$$

Equation 17 can be used along with Equation 13 to calculate σ_t as shown in Equation 14. Once again, it is worth noting that the point of truncation k_L again uniquely and solely determines the standard deviation of the standardized, left-truncated normal distribution.

Finally, it is convenient to define a coefficient of variation c – which, again, exists uniquely for a particular k_L

$$c = \frac{\sigma_t}{\mu_t} \tag{18}$$

Development of Tables of the Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution

Given the formulation of the probability density function $f_{SLTN}(t)$ of the standardized, left-truncated normal distribution in Equation 7, its cumulative distribution function $F_{SLTN}(t)$ can be stated as

$$F_{SLTN}(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{F(t + k_L) - F(k_L)}{H(k_L)}, & t \ge 0 \end{cases} \qquad (19)$$

where H(k) is as defined in Equation 12 and F(z) is the cumulative probability associated with a standard normal variate of value z, with H(z) = 1 – F(z).

Thus, given a point of truncation k_L, one can readily calculate the value of the cumulative distribution function of the standardized, left-truncated normal distribution at any standardized value of $t \ge 0$ through the use of Equation 19 and tables of the cumulative distribution function of the standard normal distribution F(z).

With the wide availability of desktop micro-computers and their associated software, alternate methods of evaluating Equation 19 at specific values of t include the use of “standard” spreadsheet functions – e.g., NormSDist(Z) in Microsoft Excel – or formulations such as those provided in [Abr72]. In developing the tables presented below, the results of utilizing the Excel worksheet functions were compared with [Abr72]'s smallest error formulation: ($|\varepsilon(x)| < 7.5 \times 10^{-8}$)

$$F(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) \qquad (20)$$

where

$$t = \frac{1}{1 + px}$$

$$p = 0.2316419,\ b_1 = 0.319381530,\ b_2 = -0.356563782,$$

$$b_3 = 1.781477937,\ b_4 = -1.821255978,\ b_5 = 1.330274429$$

In general, it was found that the results of the Excel-based calculations and those based upon Equation 20 were close enough that either could be used for purposes of developing the tables presented in this paper. Specifically, the values of the cumulative distribution function calculated using the Excel worksheet functions agreed with those tabulated in [Abr72] to at least six decimal places. Even after the computations involved with the most heavily-truncated case presented in this paper (i.e., $k_L = 3.0$), Excel's results agreed with those obtained using the tabulated values in [Abr72] to at least four decimal places. Based upon its ease of implementation, the Excel-based approach was used – with the results/values being shown to four decimal places.

Using the approach outlined above, a table of the cumulative distribution function $F_{SLTN}(t)$ for the standardized, left-truncated normal distribution as a function of the truncation point k_L has been developed. Table 1, entitled “Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution,” is presented in five, overlapping segments divided by point of truncation – with the range and granularity of standardized t’s over which $F_{SLTN}(t)$ is evaluated tailored to best illustrate the properties of the cumulative distribution function.

Using Table 1 requires only knowledge of the point of truncation k_L (in normalized terms) and, for a given point of truncation, can be utilized in a fashion comparable to other cumulative distribution tables for the (non-truncated) standard normal distribution. Detailed examples of the use of Table 1 will be presented at the end of this paper.

Figures 4 through 6 illustrate the cumulative distribution function of the standardized, left-truncated normal distribution function for seven values of the point of truncation (k_L = -3, -2, -1, 0, 1, 2, and 3) – over ranges of standardize t’s of 0 to 6, 0 to 3, and 0 to 1.5, respectively.

The "inverse" of the problem of identifying the value of the cumulative distribution $F_{SLTN}(t)$ at a standardized value t – for a given point of truncation k_L – is that of identifying the standardized value t at which the cumulative distribution function $F_{SLTN}(t)$ assumes some value. While Table 1 can, in theory, be used for such purposes – subject to the granularity of the tabulated $F_{SLTN}(t)$ values – this problem has been solved computationally through the use of a dichotomous line search algorithm, and the results are presented in Table 2 over the range $F_{SLTN}(t) = 0.005$ to $F_{SLTN}(t) = 0.995$, with varying granularity. Table 2 is presented in two segments – for negative and positive values, respectively, of the point of truncation k_L over the range -3.0(0.2)3.0.

Table 1. Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution

t/k_L	-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0012	0.0021	0.0036	0.0058	0.0090	0.0135	0.0196	0.0275	0.0373	0.0493	0.0632
0.4	0.0033	0.0057	0.0093	0.0147	0.0223	0.0328	0.0465	0.0638	0.0847	0.1094	0.1374
0.6	0.0069	0.0114	0.0182	0.0280	0.0415	0.0594	0.0821	0.1099	0.1426	0.1799	0.2210
0.8	0.0126	0.0202	0.0314	0.0470	0.0678	0.0945	0.1273	0.1662	0.2105	0.2594	0.3115
1.0	0.0214	0.0335	0.0504	0.0732	0.1026	0.1391	0.1825	0.2322	0.2870	0.3454	0.4057
1.2	0.0346	0.0524	0.0765	0.1078	0.1468	0.1935	0.2472	0.3066	0.3699	0.4350	0.4999
1.4	0.0535	0.0784	0.1109	0.1517	0.2007	0.2574	0.3202	0.3872	0.4561	0.5245	0.5904
1.6	0.0795	0.1128	0.1547	0.2053	0.2640	0.3293	0.3992	0.4710	0.5423	0.6106	0.6740
1.8	0.1139	0.1565	0.2082	0.2683	0.3353	0.4073	0.4814	0.5549	0.6252	0.6901	0.7482
2.0	0.1575	0.2098	0.2709	0.3392	0.4126	0.4884	0.5636	0.6354	0.7017	0.7606	0.8114
2.2	0.2108	0.2724	0.3415	0.4160	0.4930	0.5695	0.6426	0.7098	0.7695	0.8207	0.8632
2.4	0.2733	0.3429	0.4180	0.4959	0.5733	0.6474	0.7155	0.7759	0.8274	0.8700	0.9040
2.6	0.3437	0.4193	0.4977	0.5758	0.6506	0.7194	0.7802	0.8321	0.8748	0.9087	0.9349
2.8	0.4200	0.4987	0.5773	0.6526	0.7219	0.7832	0.8354	0.8783	0.9121	0.9381	0.9573
3.0	0.4993	0.5782	0.6538	0.7235	0.7852	0.8377	0.8806	0.9146	0.9404	0.9594	0.9730
3.2	0.5787	0.6545	0.7245	0.7864	0.8391	0.8823	0.9162	0.9420	0.9609	0.9743	0.9835
3.4	0.6550	0.7250	0.7872	0.8400	0.8833	0.9174	0.9432	0.9620	0.9753	0.9843	0.9903
3.6	0.7254	0.7876	0.8406	0.8840	0.9181	0.9439	0.9627	0.9759	0.9849	0.9907	0.9945
3.8	0.7879	0.8409	0.8844	0.9186	0.9444	0.9632	0.9764	0.9853	0.9911	0.9947	0.9970
4.0	0.8411	0.8846	0.9189	0.9447	0.9636	0.9767	0.9856	0.9913	0.9949	0.9971	0.9984
4.2	0.8848	0.9190	0.9449	0.9638	0.9769	0.9858	0.9915	0.9951	0.9972	0.9985	0.9992
4.4	0.9191	0.9451	0.9639	0.9771	0.9859	0.9916	0.9952	0.9973	0.9985	0.9992	0.9996
4.6	0.9451	0.9640	0.9771	0.9860	0.9917	0.9952	0.9973	0.9986	0.9993	0.9996	0.9998
4.8	0.9640	0.9772	0.9860	0.9917	0.9953	0.9974	0.9986	0.9993	0.9996	0.9998	0.9999
5.0	0.9772	0.9861	0.9918	0.9953	0.9974	0.9986	0.9993	0.9996	0.9998	0.9999	1.0000
5.2	0.9861	0.9918	0.9953	0.9974	0.9986	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000
5.4	0.9918	0.9953	0.9974	0.9986	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000
5.6	0.9953	0.9974	0.9986	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000
5.8	0.9974	0.9986	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
6.0	0.9986	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6.2	0.9993	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6.4	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6.6	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6.8	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1. (continued)

t/k_L	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0061	0.0090	0.0127	0.0175	0.0233	0.0302	0.0382	0.0472	0.0572	0.0681	0.0797
0.2	0.0135	0.0196	0.0275	0.0373	0.0493	0.0632	0.0792	0.0969	0.1162	0.1368	0.1585
0.3	0.0223	0.0320	0.0444	0.0597	0.0780	0.0990	0.1227	0.1486	0.1764	0.2056	0.2358
0.4	0.0328	0.0465	0.0638	0.0847	0.1094	0.1374	0.1684	0.2018	0.2371	0.2737	0.3108
0.5	0.0451	0.0631	0.0856	0.1124	0.1434	0.1781	0.2160	0.2562	0.2979	0.3404	0.3829
0.6	0.0594	0.0821	0.1099	0.1426	0.1799	0.2210	0.2650	0.3111	0.3581	0.4051	0.4515
0.7	0.0758	0.1035	0.1368	0.1754	0.2186	0.2656	0.3151	0.3659	0.4170	0.4674	0.5161
0.8	0.0945	0.1273	0.1662	0.2105	0.2594	0.3115	0.3656	0.4203	0.4743	0.5265	0.5763
0.9	0.1155	0.1537	0.1980	0.2478	0.3017	0.3584	0.4161	0.4735	0.5293	0.5823	0.6319
1.0	0.1391	0.1825	0.2322	0.2870	0.3454	0.4057	0.4662	0.5252	0.5816	0.6343	0.6827
1.1	0.1651	0.2137	0.2684	0.3278	0.3900	0.4531	0.5152	0.5749	0.6308	0.6822	0.7287
1.2	0.1935	0.2472	0.3066	0.3699	0.4350	0.4999	0.5628	0.6221	0.6768	0.7261	0.7699
1.3	0.2243	0.2828	0.3463	0.4127	0.4800	0.5459	0.6085	0.6666	0.7192	0.7658	0.8064
1.4	0.2574	0.3202	0.3872	0.4561	0.5245	0.5904	0.6520	0.7081	0.7579	0.8014	0.8385
1.5	0.2924	0.3591	0.4289	0.4994	0.5682	0.6333	0.6930	0.7464	0.7930	0.8329	0.8664
1.6	0.3293	0.3992	0.4710	0.5423	0.6106	0.6740	0.7312	0.7814	0.8244	0.8606	0.8904
1.7	0.3677	0.4401	0.5131	0.5843	0.6513	0.7124	0.7665	0.8131	0.8523	0.8847	0.9109
1.8	0.4073	0.4814	0.5549	0.6252	0.6901	0.7482	0.7987	0.8414	0.8768	0.9054	0.9281
1.9	0.4476	0.5227	0.5958	0.6644	0.7266	0.7812	0.8279	0.8666	0.8981	0.9231	0.9426
2.0	0.4884	0.5636	0.6354	0.7017	0.7606	0.8114	0.8540	0.8887	0.9164	0.9380	0.9545
2.1	0.5291	0.6037	0.6736	0.7368	0.7920	0.8388	0.8772	0.9079	0.9320	0.9504	0.9643
2.2	0.5695	0.6426	0.7098	0.7695	0.8207	0.8632	0.8975	0.9245	0.9452	0.9607	0.9722
2.3	0.6090	0.6800	0.7440	0.7998	0.8467	0.8849	0.9152	0.9386	0.9562	0.9692	0.9786
2.4	0.6474	0.7155	0.7759	0.8274	0.8700	0.9040	0.9305	0.9505	0.9653	0.9760	0.9836
2.5	0.6843	0.7490	0.8053	0.8524	0.8906	0.9206	0.9435	0.9604	0.9727	0.9815	0.9876
2.6	0.7194	0.7802	0.8321	0.8748	0.9087	0.9349	0.9544	0.9687	0.9788	0.9858	0.9907
2.7	0.7524	0.8091	0.8565	0.8947	0.9245	0.9470	0.9636	0.9754	0.9836	0.9893	0.9931
2.8	0.7832	0.8354	0.8783	0.9121	0.9381	0.9573	0.9711	0.9808	0.9875	0.9920	0.9949
2.9	0.8117	0.8593	0.8976	0.9273	0.9496	0.9659	0.9773	0.9852	0.9905	0.9940	0.9963
3.0	0.8377	0.8806	0.9146	0.9404	0.9594	0.9730	0.9824	0.9887	0.9929	0.9956	0.9973
3.1	0.8612	0.8996	0.9293	0.9515	0.9675	0.9788	0.9864	0.9914	0.9947	0.9968	0.9981
3.2	0.8823	0.9162	0.9420	0.9609	0.9743	0.9835	0.9896	0.9936	0.9961	0.9977	0.9986
3.3	0.9009	0.9307	0.9529	0.9688	0.9798	0.9873	0.9921	0.9952	0.9972	0.9983	0.9990
3.4	0.9174	0.9432	0.9620	0.9753	0.9843	0.9903	0.9941	0.9965	0.9979	0.9988	0.9993
3.5	0.9316	0.9538	0.9696	0.9806	0.9879	0.9926	0.9956	0.9974	0.9985	0.9992	0.9995
3.6	0.9439	0.9627	0.9759	0.9849	0.9907	0.9945	0.9968	0.9981	0.9990	0.9994	0.9997
3.7	0.9544	0.9702	0.9811	0.9883	0.9930	0.9959	0.9976	0.9987	0.9993	0.9996	0.9998
3.8	0.9632	0.9764	0.9853	0.9911	0.9947	0.9970	0.9983	0.9991	0.9995	0.9997	0.9999
3.9	0.9706	0.9815	0.9887	0.9932	0.9961	0.9978	0.9988	0.9993	0.9996	0.9998	0.9999
4.0	0.9767	0.9856	0.9913	0.9949	0.9971	0.9984	0.9991	0.9995	0.9998	0.9999	0.9999

Table 1. (continued)

t/k_L	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0147	0.0187	0.0233	0.0284	0.0339	0.0399	0.0462	0.0529	0.0598	0.0670	0.0744
0.10	0.0302	0.0382	0.0472	0.0572	0.0681	0.0797	0.0919	0.1046	0.1177	0.1312	0.1449
0.15	0.0464	0.0584	0.0718	0.0865	0.1024	0.1192	0.1368	0.1550	0.1737	0.1926	0.2117
0.20	0.0632	0.0792	0.0969	0.1162	0.1368	0.1585	0.1810	0.2041	0.2275	0.2511	0.2747
0.25	0.0808	0.1006	0.1225	0.1462	0.1713	0.1974	0.2243	0.2517	0.2793	0.3068	0.3341
0.30	0.0990	0.1227	0.1486	0.1764	0.2056	0.2358	0.2667	0.2978	0.3289	0.3596	0.3899
0.35	0.1179	0.1453	0.1750	0.2067	0.2397	0.2737	0.3080	0.3423	0.3763	0.4096	0.4421
0.40	0.1374	0.1684	0.2018	0.2371	0.2737	0.3108	0.3482	0.3852	0.4215	0.4568	0.4910
0.45	0.1575	0.1920	0.2289	0.2676	0.3072	0.3473	0.3872	0.4264	0.4645	0.5013	0.5365
0.50	0.1781	0.2160	0.2562	0.2979	0.3404	0.3829	0.4249	0.4658	0.5053	0.5431	0.5789
0.55	0.1993	0.2404	0.2836	0.3281	0.3730	0.4177	0.4614	0.5036	0.5440	0.5822	0.6182
0.60	0.2210	0.2650	0.3111	0.3581	0.4051	0.4515	0.4965	0.5396	0.5804	0.6188	0.6546
0.65	0.2431	0.2900	0.3385	0.3877	0.4366	0.4843	0.5302	0.5738	0.6148	0.6529	0.6882
0.70	0.2656	0.3151	0.3659	0.4170	0.4674	0.5161	0.5625	0.6063	0.6470	0.6847	0.7191
0.75	0.2884	0.3403	0.3932	0.4459	0.4974	0.5467	0.5934	0.6370	0.6773	0.7141	0.7475
0.80	0.3115	0.3656	0.4203	0.4743	0.5265	0.5763	0.6229	0.6661	0.7055	0.7413	0.7735
0.85	0.3349	0.3909	0.4471	0.5021	0.5549	0.6047	0.6510	0.6934	0.7319	0.7665	0.7973
0.90	0.3584	0.4161	0.4735	0.5293	0.5823	0.6319	0.6776	0.7191	0.7564	0.7896	0.8190
0.95	0.3820	0.4412	0.4996	0.5558	0.6088	0.6579	0.7027	0.7431	0.7791	0.8109	0.8387
1.00	0.4057	0.4662	0.5252	0.5816	0.6343	0.6827	0.7265	0.7656	0.8002	0.8304	0.8566
1.05	0.4294	0.4908	0.5503	0.6066	0.6588	0.7063	0.7489	0.7866	0.8196	0.8482	0.8728
1.10	0.4531	0.5152	0.5749	0.6308	0.6822	0.7287	0.7699	0.8061	0.8375	0.8645	0.8874
1.15	0.4766	0.5392	0.5988	0.6542	0.7047	0.7499	0.7896	0.8242	0.8539	0.8792	0.9006
1.20	0.4999	0.5628	0.6221	0.6768	0.7261	0.7699	0.8081	0.8410	0.8690	0.8926	0.9124
1.25	0.5230	0.5859	0.6447	0.6984	0.7465	0.7887	0.8252	0.8564	0.8827	0.9047	0.9229
1.30	0.5459	0.6085	0.6666	0.7192	0.7658	0.8064	0.8412	0.8707	0.8953	0.9157	0.9324
1.35	0.5683	0.6306	0.6877	0.7390	0.7841	0.8230	0.8560	0.8837	0.9067	0.9255	0.9408
1.40	0.5904	0.6520	0.7081	0.7579	0.8014	0.8385	0.8698	0.8957	0.9170	0.9344	0.9483
1.45	0.6121	0.6728	0.7276	0.7759	0.8176	0.8529	0.8824	0.9067	0.9264	0.9423	0.9550
1.50	0.6333	0.6930	0.7464	0.7930	0.8329	0.8664	0.8941	0.9167	0.9349	0.9494	0.9609
1.55	0.6539	0.7125	0.7643	0.8092	0.8472	0.8789	0.9048	0.9257	0.9425	0.9557	0.9661
1.60	0.6740	0.7312	0.7814	0.8244	0.8606	0.8904	0.9146	0.9340	0.9493	0.9613	0.9706
1.65	0.6935	0.7492	0.7976	0.8388	0.8731	0.9011	0.9236	0.9414	0.9554	0.9663	0.9746
1.70	0.7124	0.7665	0.8131	0.8523	0.8847	0.9109	0.9317	0.9482	0.9609	0.9707	0.9781
1.75	0.7306	0.7830	0.8277	0.8650	0.8954	0.9199	0.9392	0.9542	0.9658	0.9746	0.9812
1.80	0.7482	0.7987	0.8414	0.8768	0.9054	0.9281	0.9459	0.9597	0.9701	0.9780	0.9839
1.85	0.7651	0.8137	0.8544	0.8878	0.9146	0.9357	0.9520	0.9645	0.9740	0.9810	0.9862
1.90	0.7812	0.8279	0.8666	0.8981	0.9231	0.9426	0.9575	0.9689	0.9774	0.9836	0.9882
1.95	0.7967	0.8413	0.8780	0.9076	0.9308	0.9488	0.9625	0.9728	0.9804	0.9859	0.9900
2.00	0.8114	0.8540	0.8887	0.9164	0.9380	0.9545	0.9670	0.9762	0.9830	0.9879	0.9915
2.05	0.8254	0.8660	0.8987	0.9245	0.9445	0.9596	0.9709	0.9793	0.9853	0.9897	0.9928
2.10	0.8388	0.8772	0.9079	0.9320	0.9504	0.9643	0.9745	0.9820	0.9874	0.9912	0.9939
2.15	0.8513	0.8877	0.9165	0.9389	0.9558	0.9684	0.9777	0.9844	0.9891	0.9925	0.9949
2.20	0.8632	0.8975	0.9245	0.9452	0.9607	0.9722	0.9805	0.9865	0.9907	0.9936	0.9957
2.25	0.8744	0.9067	0.9318	0.9509	0.9652	0.9756	0.9830	0.9883	0.9920	0.9946	0.9964
2.30	0.8849	0.9152	0.9386	0.9562	0.9692	0.9786	0.9852	0.9899	0.9932	0.9954	0.9970
2.35	0.8948	0.9231	0.9448	0.9610	0.9728	0.9812	0.9872	0.9914	0.9942	0.9961	0.9975
2.40	0.9040	0.9305	0.9505	0.9653	0.9760	0.9836	0.9889	0.9926	0.9951	0.9968	0.9979
2.45	0.9126	0.9372	0.9557	0.9692	0.9789	0.9857	0.9904	0.9937	0.9958	0.9973	0.9982
2.50	0.9206	0.9435	0.9604	0.9727	0.9815	0.9876	0.9918	0.9946	0.9965	0.9977	0.9985

Table 1. (continued)

t/k_L	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.025	0.0199	0.0232	0.0266	0.0301	0.0338	0.0377	0.0416	0.0455	0.0496	0.0537	0.0579
0.050	0.0399	0.0462	0.0529	0.0598	0.0670	0.0744	0.0819	0.0895	0.0972	0.1050	0.1129
0.075	0.0598	0.0691	0.0789	0.0890	0.0994	0.1101	0.1209	0.1319	0.1429	0.1540	0.1651
0.100	0.0797	0.0919	0.1046	0.1177	0.1312	0.1449	0.1588	0.1727	0.1868	0.2008	0.2148
0.125	0.0995	0.1144	0.1300	0.1459	0.1622	0.1788	0.1954	0.2121	0.2288	0.2454	0.2618
0.150	0.1192	0.1368	0.1550	0.1737	0.1926	0.2117	0.2308	0.2500	0.2690	0.2878	0.3065
0.175	0.1389	0.1590	0.1797	0.2008	0.2222	0.2437	0.2651	0.2864	0.3075	0.3283	0.3488
0.200	0.1585	0.1810	0.2041	0.2275	0.2511	0.2747	0.2982	0.3214	0.3443	0.3668	0.3889
0.225	0.1780	0.2028	0.2281	0.2537	0.2793	0.3049	0.3302	0.3551	0.3795	0.4035	0.4268
0.250	0.1974	0.2243	0.2517	0.2793	0.3068	0.3341	0.3610	0.3874	0.4132	0.4383	0.4627
0.275	0.2167	0.2456	0.2749	0.3043	0.3336	0.3624	0.3907	0.4184	0.4453	0.4714	0.4966
0.300	0.2358	0.2667	0.2978	0.3289	0.3596	0.3899	0.4194	0.4482	0.4760	0.5028	0.5286
0.325	0.2548	0.2875	0.3203	0.3528	0.3850	0.4164	0.4470	0.4767	0.5052	0.5326	0.5589
0.350	0.2737	0.3080	0.3423	0.3763	0.4096	0.4421	0.4736	0.5040	0.5331	0.5609	0.5874
0.375	0.2923	0.3282	0.3639	0.3992	0.4336	0.4670	0.4992	0.5301	0.5596	0.5877	0.6143
0.400	0.3108	0.3482	0.3852	0.4215	0.4568	0.4910	0.5238	0.5551	0.5848	0.6130	0.6397
0.425	0.3292	0.3678	0.4060	0.4433	0.4794	0.5142	0.5474	0.5790	0.6089	0.6371	0.6636
0.450	0.3473	0.3872	0.4264	0.4645	0.5013	0.5365	0.5701	0.6018	0.6317	0.6598	0.6860
0.475	0.3652	0.4062	0.4463	0.4852	0.5225	0.5581	0.5918	0.6236	0.6534	0.6812	0.7072
0.500	0.3829	0.4249	0.4658	0.5053	0.5431	0.5789	0.6127	0.6444	0.6740	0.7015	0.7270
0.525	0.4004	0.4433	0.4849	0.5249	0.5630	0.5989	0.6327	0.6642	0.6935	0.7207	0.7457
0.550	0.4177	0.4614	0.5036	0.5440	0.5822	0.6182	0.6519	0.6831	0.7121	0.7388	0.7632
0.575	0.4347	0.4791	0.5218	0.5625	0.6008	0.6368	0.6702	0.7012	0.7296	0.7558	0.7797
0.600	0.4515	0.4965	0.5396	0.5804	0.6188	0.6546	0.6878	0.7183	0.7463	0.7718	0.7951
0.625	0.4680	0.5135	0.5569	0.5979	0.6362	0.6717	0.7045	0.7346	0.7620	0.7870	0.8096
0.650	0.4843	0.5302	0.5738	0.6148	0.6529	0.6882	0.7205	0.7501	0.7769	0.8012	0.8231
0.675	0.5003	0.5465	0.5903	0.6312	0.6691	0.7040	0.7358	0.7648	0.7910	0.8146	0.8358
0.700	0.5161	0.5625	0.6063	0.6470	0.6847	0.7191	0.7504	0.7788	0.8043	0.8272	0.8476
0.725	0.5315	0.5782	0.6219	0.6624	0.6997	0.7336	0.7644	0.7921	0.8169	0.8390	0.8587
0.750	0.5467	0.5934	0.6370	0.6773	0.7141	0.7475	0.7776	0.8046	0.8287	0.8501	0.8690
0.775	0.5617	0.6084	0.6518	0.6917	0.7280	0.7608	0.7903	0.8165	0.8399	0.8605	0.8787
0.800	0.5763	0.6229	0.6661	0.7055	0.7413	0.7735	0.8023	0.8278	0.8504	0.8703	0.8877
0.825	0.5906	0.6371	0.6799	0.7190	0.7542	0.7857	0.8137	0.8385	0.8603	0.8794	0.8961
0.850	0.6047	0.6510	0.6934	0.7319	0.7665	0.7973	0.8246	0.8486	0.8697	0.8880	0.9039
0.875	0.6184	0.6644	0.7064	0.7444	0.7783	0.8084	0.8349	0.8582	0.8784	0.8960	0.9112
0.900	0.6319	0.6776	0.7191	0.7564	0.7896	0.8190	0.8448	0.8672	0.8867	0.9035	0.9180
0.925	0.6450	0.6903	0.7313	0.7680	0.8005	0.8291	0.8541	0.8757	0.8944	0.9105	0.9243
0.950	0.6579	0.7027	0.7431	0.7791	0.8109	0.8387	0.8629	0.8838	0.9017	0.9171	0.9302
0.975	0.6704	0.7148	0.7546	0.7899	0.8209	0.8479	0.8713	0.8913	0.9085	0.9232	0.9356
1.000	0.6827	0.7265	0.7656	0.8002	0.8304	0.8566	0.8792	0.8985	0.9149	0.9289	0.9407
1.025	0.6946	0.7379	0.7763	0.8101	0.8395	0.8649	0.8867	0.9052	0.9209	0.9342	0.9454
1.050	0.7063	0.7489	0.7866	0.8196	0.8482	0.8728	0.8938	0.9116	0.9266	0.9392	0.9497
1.075	0.7176	0.7596	0.7965	0.8287	0.8565	0.8803	0.9005	0.9175	0.9318	0.9438	0.9537
1.100	0.7287	0.7699	0.8061	0.8375	0.8645	0.8874	0.9068	0.9231	0.9367	0.9481	0.9575
1.125	0.7394	0.7799	0.8153	0.8459	0.8720	0.8942	0.9128	0.9284	0.9413	0.9521	0.9609
1.150	0.7499	0.7896	0.8242	0.8539	0.8792	0.9006	0.9184	0.9333	0.9456	0.9558	0.9641
1.175	0.7600	0.7990	0.8328	0.8616	0.8861	0.9066	0.9237	0.9379	0.9496	0.9592	0.9671
1.200	0.7699	0.8081	0.8410	0.8690	0.8926	0.9124	0.9288	0.9423	0.9534	0.9624	0.9698
1.225	0.7794	0.8168	0.8489	0.8760	0.8988	0.9178	0.9335	0.9464	0.9569	0.9654	0.9723
1.250	0.7887	0.8252	0.8564	0.8827	0.9047	0.9229	0.9379	0.9502	0.9601	0.9682	0.9746

Table 1. (continued)

t/k_L	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.015	0.0227	0.0251	0.0275	0.0300	0.0325	0.0351	0.0376	0.0402	0.0429	0.0455	0.0482
0.030	0.0451	0.0497	0.0545	0.0593	0.0642	0.0691	0.0741	0.0791	0.0841	0.0891	0.0942
0.045	0.0671	0.0739	0.0808	0.0879	0.0949	0.1021	0.1093	0.1165	0.1237	0.1309	0.1381
0.060	0.0888	0.0976	0.1066	0.1157	0.1249	0.1341	0.1433	0.1526	0.1618	0.1710	0.1802
0.075	0.1101	0.1209	0.1319	0.1429	0.1540	0.1651	0.1762	0.1873	0.1984	0.2094	0.2203
0.090	0.1311	0.1438	0.1566	0.1694	0.1823	0.1952	0.2081	0.2208	0.2335	0.2461	0.2586
0.105	0.1517	0.1662	0.1807	0.1953	0.2099	0.2244	0.2388	0.2531	0.2673	0.2813	0.2952
0.120	0.1721	0.1882	0.2043	0.2205	0.2366	0.2526	0.2685	0.2842	0.2997	0.3150	0.3301
0.135	0.1920	0.2097	0.2274	0.2451	0.2626	0.2800	0.2972	0.3141	0.3308	0.3473	0.3634
0.150	0.2117	0.2308	0.2500	0.2690	0.2878	0.3065	0.3249	0.3430	0.3607	0.3782	0.3952
0.165	0.2310	0.2515	0.2720	0.2923	0.3124	0.3321	0.3516	0.3707	0.3894	0.4077	0.4256
0.180	0.2499	0.2718	0.2935	0.3150	0.3362	0.3570	0.3774	0.3974	0.4169	0.4359	0.4545
0.195	0.2686	0.2917	0.3145	0.3371	0.3593	0.3810	0.4023	0.4230	0.4433	0.4629	0.4820
0.210	0.2869	0.3111	0.3351	0.3586	0.3817	0.4043	0.4263	0.4477	0.4686	0.4888	0.5083
0.225	0.3049	0.3302	0.3551	0.3795	0.4035	0.4268	0.4495	0.4715	0.4928	0.5134	0.5334
0.240	0.3225	0.3488	0.3746	0.3999	0.4246	0.4486	0.4718	0.4943	0.5161	0.5370	0.5572
0.255	0.3398	0.3670	0.3937	0.4197	0.4451	0.4696	0.4933	0.5163	0.5383	0.5596	0.5800
0.270	0.3568	0.3849	0.4123	0.4390	0.4649	0.4899	0.5141	0.5373	0.5597	0.5811	0.6016
0.285	0.3735	0.4023	0.4305	0.4578	0.4842	0.5096	0.5341	0.5576	0.5801	0.6017	0.6222
0.300	0.3899	0.4194	0.4482	0.4760	0.5028	0.5286	0.5534	0.5771	0.5997	0.6213	0.6419
0.315	0.4059	0.4361	0.4654	0.4937	0.5209	0.5470	0.5719	0.5958	0.6184	0.6400	0.6605
0.330	0.4216	0.4524	0.4822	0.5109	0.5384	0.5647	0.5898	0.6137	0.6364	0.6579	0.6783
0.345	0.4371	0.4684	0.4986	0.5276	0.5554	0.5818	0.6070	0.6309	0.6536	0.6750	0.6952
0.360	0.4522	0.4840	0.5145	0.5438	0.5718	0.5984	0.6236	0.6474	0.6700	0.6913	0.7113
0.375	0.4670	0.4992	0.5301	0.5596	0.5877	0.6143	0.6395	0.6633	0.6857	0.7068	0.7266
0.390	0.4815	0.5141	0.5452	0.5749	0.6031	0.6297	0.6548	0.6785	0.7007	0.7216	0.7411
0.405	0.4957	0.5286	0.5599	0.5897	0.6180	0.6446	0.6696	0.6931	0.7151	0.7357	0.7549
0.420	0.5096	0.5427	0.5743	0.6042	0.6324	0.6589	0.6838	0.7071	0.7288	0.7491	0.7680
0.435	0.5232	0.5566	0.5882	0.6181	0.6463	0.6727	0.6974	0.7205	0.7420	0.7619	0.7805
0.450	0.5365	0.5701	0.6018	0.6317	0.6598	0.6860	0.7105	0.7333	0.7545	0.7742	0.7923
0.465	0.5496	0.5832	0.6150	0.6449	0.6728	0.6989	0.7231	0.7456	0.7665	0.7858	0.8036
0.480	0.5623	0.5961	0.6278	0.6576	0.6854	0.7112	0.7352	0.7574	0.7779	0.7968	0.8143
0.495	0.5748	0.6086	0.6403	0.6700	0.6976	0.7232	0.7469	0.7687	0.7889	0.8074	0.8244
0.510	0.5870	0.6208	0.6525	0.6819	0.7093	0.7347	0.7580	0.7795	0.7993	0.8174	0.8340
0.525	0.5989	0.6327	0.6642	0.6935	0.7207	0.7457	0.7688	0.7899	0.8093	0.8270	0.8431
0.540	0.6106	0.6443	0.6757	0.7048	0.7317	0.7564	0.7790	0.7998	0.8188	0.8360	0.8518
0.555	0.6220	0.6556	0.6868	0.7157	0.7422	0.7666	0.7889	0.8093	0.8278	0.8447	0.8600
0.570	0.6331	0.6666	0.6976	0.7262	0.7525	0.7765	0.7984	0.8184	0.8365	0.8529	0.8678
0.585	0.6440	0.6773	0.7081	0.7364	0.7623	0.7860	0.8075	0.8270	0.8447	0.8607	0.8751
0.600	0.6546	0.6878	0.7183	0.7463	0.7718	0.7951	0.8162	0.8353	0.8526	0.8681	0.8821
0.615	0.6650	0.6979	0.7282	0.7558	0.7810	0.8039	0.8246	0.8432	0.8601	0.8752	0.8887
0.630	0.6751	0.7078	0.7378	0.7651	0.7899	0.8123	0.8326	0.8508	0.8672	0.8819	0.8950
0.645	0.6849	0.7174	0.7471	0.7740	0.7984	0.8205	0.8403	0.8581	0.8740	0.8882	0.9009
0.660	0.6946	0.7268	0.7561	0.7827	0.8067	0.8283	0.8476	0.8650	0.8805	0.8943	0.9066
0.675	0.7040	0.7358	0.7648	0.7910	0.8146	0.8358	0.8547	0.8716	0.8866	0.9000	0.9119
0.690	0.7131	0.7447	0.7733	0.7991	0.8222	0.8430	0.8615	0.8779	0.8925	0.9055	0.9169
0.705	0.7221	0.7533	0.7815	0.8069	0.8296	0.8499	0.8679	0.8839	0.8981	0.9106	0.9217
0.720	0.7308	0.7616	0.7895	0.8144	0.8367	0.8565	0.8741	0.8897	0.9034	0.9155	0.9262
0.735	0.7392	0.7697	0.7972	0.8217	0.8435	0.8629	0.8800	0.8952	0.9085	0.9202	0.9305
0.750	0.7475	0.7776	0.8046	0.8287	0.8501	0.8690	0.8857	0.9004	0.9133	0.9246	0.9345

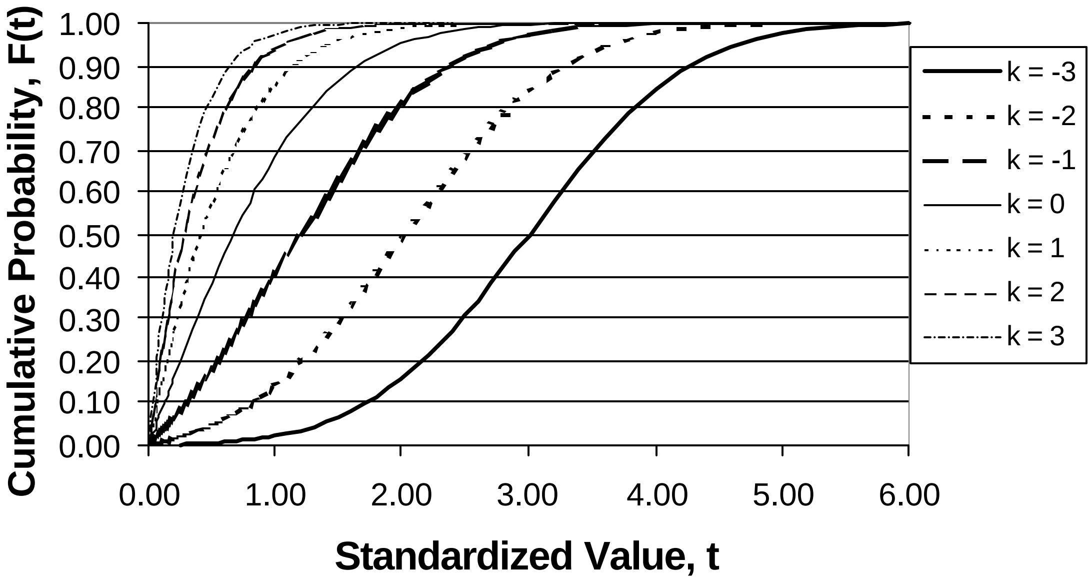

Figure 4. Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution for Standardized t's of 0 to 6

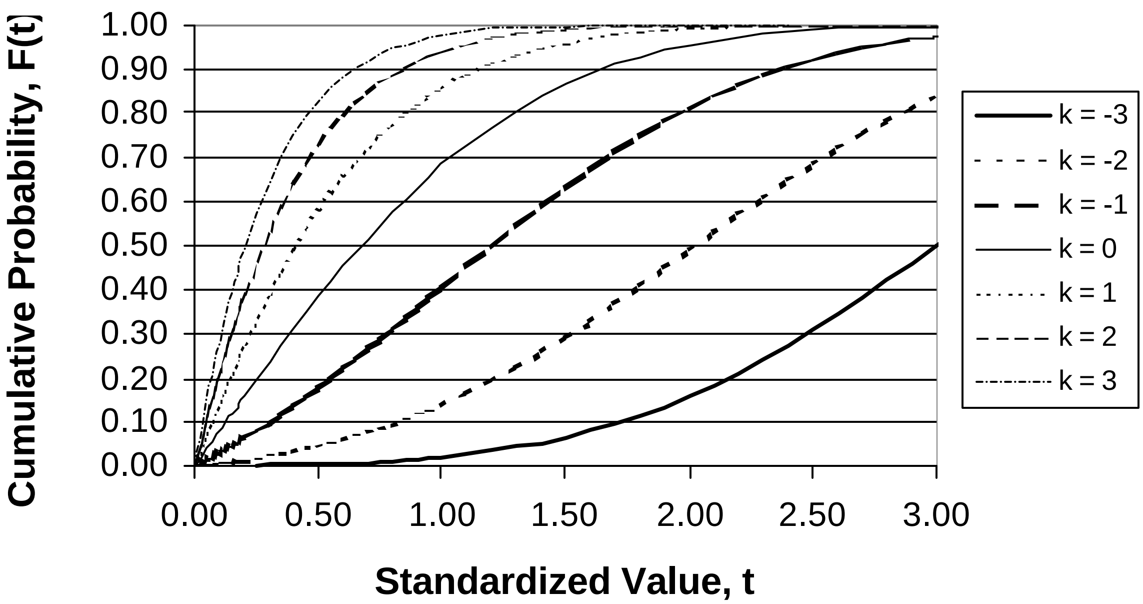

Figure 5. Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution for Standardized t's of 0 to 3

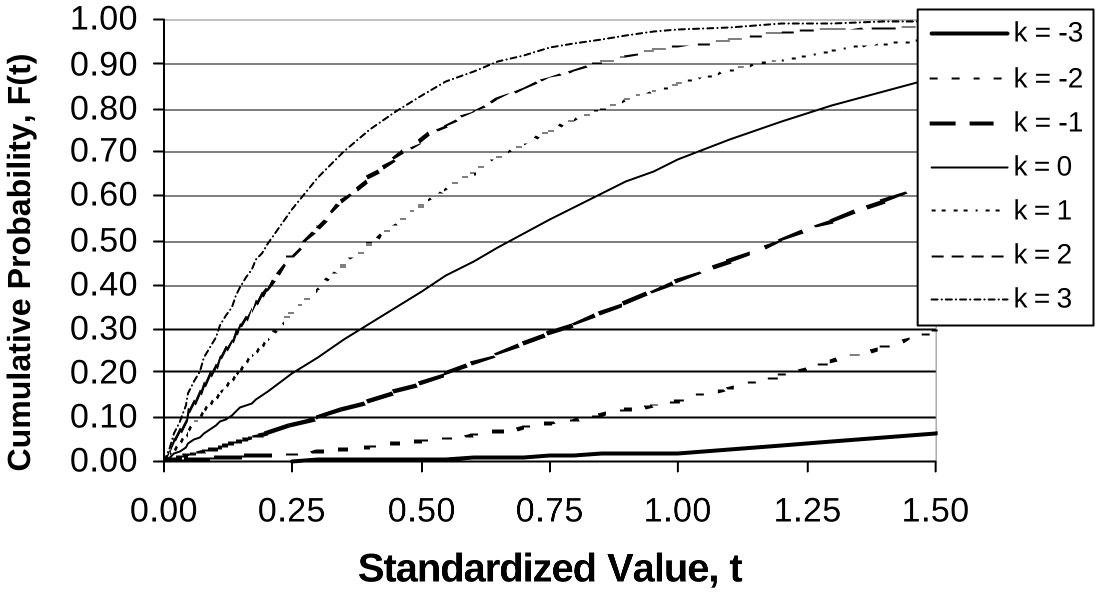

Figure 6. Cumulative Distribution Function of the Standardized, Left-Truncated Normal Distribution for Standardized t’s of 0 to 1.5

Table 2. Standard t Value Associated with a Cumulative Distribution Value F(t) For the Standardized, Left-Truncated Normal Distribution

$F(t)/k_L$	-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
0.005	0.5075	0.3697	0.2598	0.1784	0.1215	0.0833	0.0580	0.0412	0.0301	0.0225	0.0172	0.0135	0.0109	0.0089	0.0074	0.0063
0.010	0.7211	0.5595	0.4196	0.3057	0.2184	0.1550	0.1106	0.0800	0.0589	0.0444	0.0342	0.0269	0.0216	0.0177	0.0148	0.0125
0.020	0.9728	0.7954	0.6326	0.4895	0.3700	0.2753	0.2037	0.1512	0.1135	0.0866	0.0673	0.0533	0.0430	0.0353	0.0295	0.0251
0.025	1.0621	0.8810	0.7125	0.5615	0.4323	0.3272	0.2455	0.1843	0.1395	0.1070	0.0834	0.0663	0.0536	0.0441	0.0369	0.0313
0.030	1.1381	0.9545	0.7819	0.6250	0.4884	0.3749	0.2847	0.2158	0.1646	0.1269	0.0994	0.0791	0.0641	0.0529	0.0443	0.0376
0.040	1.2642	1.0771	0.8990	0.7341	0.5868	0.4605	0.3569	0.2751	0.2126	0.1656	0.1306	0.1045	0.0850	0.0702	0.0589	0.0502
0.050	1.3675	1.1782	0.9966	0.8264	0.6716	0.5361	0.4222	0.3301	0.2580	0.2027	0.1610	0.1295	0.1057	0.0876	0.0736	0.0627
0.075	1.5692	1.3770	1.1903	1.0121	0.8460	0.6956	0.5640	0.4530	0.3623	0.2901	0.2337	0.1901	0.1564	0.1304	0.1101	0.0941
0.100	1.7253	1.5314	1.3420	1.1594	0.9867	0.8274	0.6846	0.5606	0.4564	0.3710	0.3026	0.2486	0.2061	0.1728	0.1465	0.1257
0.150	1.9685	1.7728	1.5804	1.3930	1.2130	1.0432	0.8867	0.7461	0.6232	0.5186	0.4315	0.3603	0.3027	0.2564	0.2191	0.1891
0.200	2.1622	1.9657	1.7716	1.5816	1.3975	1.2217	1.0570	0.9060	0.7708	0.6526	0.5516	0.4668	0.3966	0.3390	0.2919	0.2534
0.250	2.3287	2.1315	1.9365	1.7447	1.5580	1.3783	1.2081	1.0498	0.9058	0.7774	0.6655	0.5696	0.4887	0.4211	0.3651	0.3186
0.300	2.4783	2.2807	2.0850	1.8920	1.7034	1.5209	1.3467	1.1831	1.0322	0.8960	0.7752	0.6700	0.5799	0.5035	0.4392	0.3853
0.400	2.7487	2.5506	2.3539	2.1594	1.9682	1.7818	1.6021	1.4310	1.2704	1.1223	0.9879	0.8680	0.7625	0.6708	0.5919	0.5244
0.500	3.0017	2.8032	2.6058	2.4103	2.2174	2.0285	1.8450	1.6687	1.5014	1.3447	1.2002	1.0687	0.9508	0.8462	0.7545	0.6745
0.600	3.2547	3.0560	2.8582	2.6618	2.4678	2.2770	2.0907	1.9105	1.7380	1.5746	1.4219	1.2810	1.1525	1.0367	0.9332	0.8416
0.700	3.5256	3.3266	3.1284	2.9315	2.7364	2.5441	2.3557	2.1723	1.9954	1.8265	1.6669	1.5178	1.3799	1.2537	1.1393	1.0364
0.750	3.6755	3.4765	3.2782	3.0810	2.8855	2.6925	2.5030	2.3183	2.1395	1.9680	1.8053	1.6523	1.5099	1.3787	1.2589	1.1504
0.800	3.8426	3.6434	3.4450	3.2475	3.0516	2.8580	2.6676	2.4814	2.3008	2.1269	1.9610	1.8043	1.6575	1.5213	1.3960	1.2816
0.850	4.0373	3.8381	3.6394	3.4417	3.2454	3.0512	2.8598	2.6724	2.4899	2.3135	2.1445	1.9839	1.8326	1.6912	1.5602	1.4395
0.900	4.2823	4.0830	3.8842	3.6862	3.4895	3.2946	3.1023	2.9134	2.7290	2.5501	2.3778	2.2131	2.0569	1.9098	1.7724	1.6449
0.925	4.4402	4.2409	4.0420	3.8439	3.6469	3.4517	3.2588	3.0692	2.8837	2.7034	2.5293	2.3623	2.2033	2.0531	1.9120	1.7805
0.950	4.6455	4.4461	4.2471	4.0488	3.8516	3.6560	3.4625	3.2720	3.0853	2.9034	2.7272	2.5576	2.3955	2.2415	2.0963	1.9600
0.960	4.7513	4.5519	4.3529	4.1545	3.9572	3.7613	3.5676	3.3767	3.1895	3.0068	2.8296	2.6588	2.4952	2.3396	2.1923	2.0537
0.970	4.8814	4.6819	4.4829	4.2844	4.0870	3.8909	3.6969	3.5055	3.3177	3.1341	2.9559	2.7837	2.6184	2.4608	2.3112	2.1701
0.975	4.9605	4.7611	4.5620	4.3635	4.1659	3.9698	3.7756	3.5840	3.3957	3.2117	3.0328	2.8599	2.6937	2.5349	2.3840	2.2414
0.980	5.0543	4.8548	4.6557	4.4571	4.2595	4.0632	3.8688	3.6769	3.4883	3.3038	3.1242	2.9504	2.7831	2.6230	2.4706	2.3264
0.990	5.3269	5.1273	4.9281	4.7294	4.5316	4.3350	4.1400	3.9474	3.7578	3.5719	3.3905	3.2144	3.0443	2.8808	2.7245	2.5758
0.995	5.5763	5.3767	5.1774	4.9787	4.7807	4.5838	4.3885	4.1953	4.0048	3.8178	3.6350	3.4571	3.2848	3.1187	2.9593	2.8070
Mean	3.0044	2.8079	2.6136	2.4226	2.2360	2.0552	1.8819	1.7174	1.5629	1.4194	1.2876	1.1676	1.0591	0.9619	0.8751	0.7979
Std Dev	0.9933	0.9888	0.9820	0.9723	0.9589	0.9415	0.9197	0.8936	0.8634	0.8298	0.7935	0.7555	0.7167	0.6779	0.6397	0.6028
CoV	0.3306	0.3521	0.3757	0.4013	0.4289	0.4581	0.4887	0.5203	0.5524	0.5846	0.6163	0.6471	0.6767	0.7048	0.7311	0.7555

Table 2. (continued)

$F(t)/k_L$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.005	0.0063	0.0054	0.0047	0.0041	0.0037	0.0033	0.0030	0.0027	0.0025	0.0023	0.0021	0.0020	0.0018	0.0017	0.0016	0.0015
0.010	0.0125	0.0108	0.0094	0.0083	0.0073	0.0066	0.0059	0.0054	0.0050	0.0046	0.0042	0.0039	0.0037	0.0034	0.0032	0.0031
0.020	0.0251	0.0216	0.0188	0.0165	0.0147	0.0132	0.0119	0.0109	0.0100	0.0092	0.0085	0.0079	0.0074	0.0069	0.0065	0.0061
0.025	0.0313	0.0270	0.0235	0.0207	0.0184	0.0165	0.0149	0.0136	0.0125	0.0115	0.0106	0.0099	0.0093	0.0087	0.0082	0.0077
0.030	0.0376	0.0324	0.0282	0.0249	0.0221	0.0199	0.0180	0.0164	0.0150	0.0138	0.0128	0.0119	0.0111	0.0104	0.0098	0.0093
0.040	0.0502	0.0432	0.0377	0.0333	0.0296	0.0266	0.0240	0.0219	0.0201	0.0185	0.0171	0.0160	0.0149	0.0140	0.0132	0.0124
0.050	0.0627	0.0541	0.0472	0.0417	0.0371	0.0333	0.0302	0.0275	0.0252	0.0232	0.0215	0.0200	0.0187	0.0176	0.0165	0.0156
0.075	0.0941	0.0814	0.0712	0.0629	0.0561	0.0504	0.0457	0.0417	0.0382	0.0352	0.0327	0.0304	0.0284	0.0266	0.0251	0.0237
0.100	0.1257	0.1090	0.0955	0.0845	0.0754	0.0679	0.0615	0.0561	0.0515	0.0475	0.0440	0.0410	0.0383	0.0360	0.0338	0.0320
0.150	0.1891	0.1648	0.1450	0.1286	0.1151	0.1037	0.0941	0.0860	0.0790	0.0729	0.0676	0.0630	0.0589	0.0553	0.0521	0.0492
0.200	0.2534	0.2218	0.1958	0.1742	0.1562	0.1410	0.1282	0.1172	0.1078	0.0996	0.0924	0.0862	0.0806	0.0757	0.0713	0.0673
0.250	0.3186	0.2802	0.2482	0.2215	0.1990	0.1800	0.1639	0.1500	0.1381	0.1277	0.1186	0.1106	0.1035	0.0972	0.0916	0.0866
0.300	0.3853	0.3402	0.3024	0.2706	0.2438	0.2209	0.2014	0.1846	0.1701	0.1574	0.1463	0.1365	0.1278	0.1201	0.1132	0.1070
0.400	0.5244	0.4668	0.4178	0.3759	0.3401	0.3094	0.2830	0.2600	0.2401	0.2226	0.2072	0.1936	0.1815	0.1707	0.1610	0.1523
0.500	0.6745	0.6051	0.5452	0.4933	0.4485	0.4096	0.3758	0.3463	0.3205	0.2977	0.2776	0.2598	0.2438	0.2296	0.2167	0.2052
0.600	0.8416	0.7609	0.6901	0.6281	0.5739	0.5263	0.4846	0.4480	0.4156	0.3870	0.3615	0.3388	0.3185	0.3002	0.2838	0.2689
0.700	1.0364	0.9444	0.8626	0.7899	0.7256	0.6686	0.6181	0.5734	0.5335	0.4981	0.4663	0.4379	0.4123	0.3893	0.3684	0.3494
0.750	1.1504	1.0525	0.9649	0.8866	0.8168	0.7546	0.6992	0.6499	0.6058	0.5663	0.5310	0.4992	0.4705	0.4445	0.4210	0.3996
0.800	1.2816	1.1777	1.0839	0.9995	0.9238	0.8559	0.7951	0.7406	0.6918	0.6478	0.6083	0.5726	0.5403	0.5110	0.4844	0.4601
0.850	1.4395	1.3292	1.2287	1.1376	1.0553	0.9810	0.9139	0.8535	0.7990	0.7498	0.7053	0.6649	0.6283	0.5950	0.5646	0.5369
0.900	1.6449	1.5271	1.4190	1.3201	1.2299	1.1478	1.0732	1.0055	0.9440	0.8881	0.8373	0.7910	0.7488	0.7103	0.6750	0.6426
0.925	1.7805	1.6584	1.5457	1.4421	1.3472	1.2604	1.1811	1.1088	1.0429	0.9828	0.9279	0.8778	0.8319	0.7899	0.7514	0.7160
0.950	1.9600	1.8328	1.7147	1.6054	1.5047	1.4120	1.3269	1.2489	1.1774	1.1119	1.0518	0.9967	0.9461	0.8996	0.8567	0.8172
0.960	2.0537	1.9241	1.8034	1.6914	1.5878	1.4923	1.4043	1.3234	1.2491	1.1809	1.1182	1.0605	1.0075	0.9586	0.9136	0.8720
0.970	2.1701	2.0376	1.9139	1.7987	1.6918	1.5928	1.5014	1.4171	1.3395	1.2679	1.2020	1.1412	1.0852	1.0335	0.9857	0.9415
0.975	2.2414	2.1073	1.9818	1.8647	1.7559	1.6549	1.5615	1.4752	1.3955	1.3220	1.2541	1.1915	1.1337	1.0803	1.0308	0.9850
0.980	2.3264	2.1904	2.0629	1.9436	1.8326	1.7293	1.6336	1.5449	1.4629	1.3871	1.3170	1.2522	1.1923	1.1368	1.0854	1.0377
0.990	2.5758	2.4350	2.3020	2.1771	2.0600	1.9505	1.8483	1.7532	1.6647	1.5824	1.5060	1.4350	1.3690	1.3077	1.2506	1.1975
0.995	2.8070	2.6622	2.5249	2.3952	2.2731	2.1584	2.0509	1.9502	1.8562	1.7684	1.6864	1.6100	1.5387	1.4721	1.4100	1.3520
Mean	0.7979	0.7294	0.6688	0.6150	0.5674	0.5251	0.4876	0.4541	0.4241	0.3973	0.3732	0.3515	0.3319	0.3140	0.2978	0.2829
Std Dev	0.6028	0.5675	0.5341	0.5027	0.4734	0.4462	0.4210	0.3977	0.3762	0.3563	0.3380	0.3211	0.3056	0.2914	0.2784	0.2668
CoV	0.7555	0.7780	0.7986	0.8174	0.8344	0.8497	0.8635	0.8759	0.8869	0.8968	0.9057	0.9136	0.9209	0.9278	0.9348	0.9428

As with Table 1, use of Table 2 requires only knowledge of the truncation point k_L. Again, detailed examples to illustrate the use of Table 2 will be presented at the end of this paper.

In addition to the standardized t values associated with different values of the cumulative distribution $F_{SLTN}(t)$ – for a given point of truncation k_L – Table 2 also identifies the mean (μ_t) and standard deviation (σ_t) associated with each truncation point. These values, which were calculated using Equations 13 and 17, respectively, were then used to calculate the coefficient of variation c associated with each truncation point. As noted earlier, this coefficient of variation is uniquely determined for the standardized, left-truncated normal distribution by the value of the truncation point k_L. As noted in [Tho80], the coefficient of variation c is approximately equal to 0.33 for light truncation ($k \sim -3$) and approaches 1.0 for heavy truncation ($k \sim +3$).

For situations in which the truncation point is not precisely known, it can be estimated using order statistics, and then Tables 1 and 2 can be utilized. Alternatively, the sample mean and standard deviation can be used to estimate the coefficient of variation as follows

$$c = \frac{\sigma_t}{\mu_t} \approx \frac{s}{\overline{x}} \tag{21}$$

where the accuracy of this approximation will improve as the sample size increases. This estimated coefficient of variation c can then be used along with Table 2 to identify the approximate standardized truncation point k_L – subject to the granularity of the coefficients of variation presented in Table 2.

To assist in such estimation of the truncation point, Table 3 summarizes the truncation point k_L associated with coefficients of variation c over the range 0.2(0.02)1.68 – which were determined using a dichotomous line search algorithm. In addition, Figure 7 plots the coefficient of variation c as a function of the point of truncation k_L.

Examples of Use of the Tables for the Standardized, Left-Truncated Normal Distribution

Table 1. An inventory manager is developing a plan for the stocking of an item whose monthly demand is normally distributed with a mean of 20 units and a standard deviation of 10 units. Specifically, he is interested in determining the probability that the monthly demand will exceed 30 units.

Table 3. Associated Values of the Coefficient of Variation c and Truncation Point k_L for the Standardized, Left-Truncated Normal Distribution

c	k_L	c	k_L	c	k_L	c	k_L	c	k_L
0.20	-5.00	0.50	-1.73	0.80	0.41	1.10	4.18	1.40	4.80
0.22	-4.55	0.52	-1.60	0.82	0.63	1.12	4.24	1.42	4.83
0.24	-4.17	0.54	-1.48	0.84	0.87	1.14	4.30	1.44	4.86
0.26	-3.84	0.56	-1.35	0.86	1.15	1.16	4.35	1.46	4.88
0.28	-3.57	0.58	-1.23	0.88	1.47	1.18	4.40	1.48	4.91
0.30	-3.32	0.60	-1.10	0.90	1.87	1.20	4.45	1.50	4.94
0.32	-3.11	0.62	-0.98	0.92	2.38	1.22	4.49	1.52	4.96
0.34	-2.91	0.64	-0.85	0.94	2.93	1.24	4.53	1.54	4.99
0.36	-2.73	0.66	-0.71	0.96	3.31	1.26	4.57	1.56	5.01
0.38	-2.57	0.68	-0.58	0.98	3.55	1.28	4.61	1.58	5.03
0.40	-2.41	0.70	-0.43	1.00	3.71	1.30	4.64	1.60	5.05
0.42	-2.26	0.72	-0.29	1.02	3.84	1.32	4.68	1.62	5.08
0.44	-2.12	0.74	-0.13	1.04	3.94	1.34	4.71	1.64	5.10
0.46	-1.99	0.76	0.04	1.06	4.03	1.36	4.74	1.66	5.12
0.48	-1.86	0.78	0.22	1.08	4.11	1.38	4.77	1.68	5.14

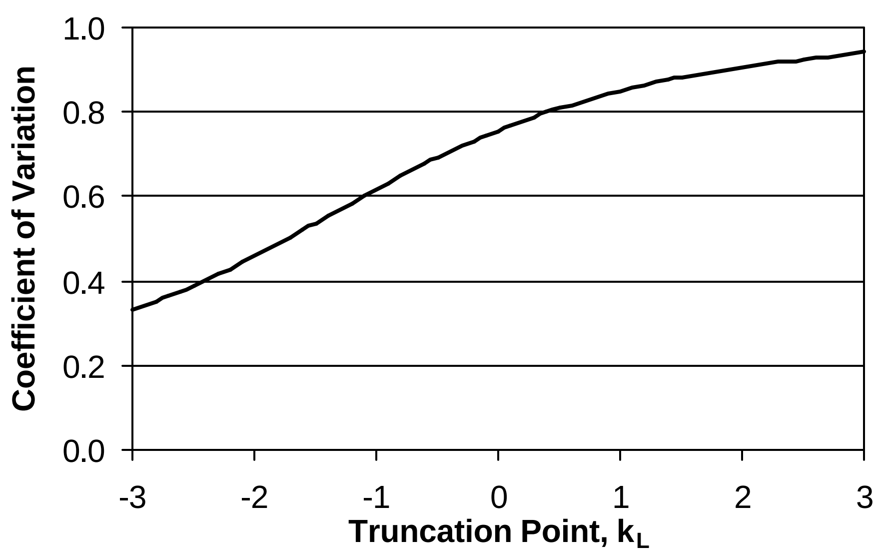

Figure 7. Coefficient of Variation as a Function of Truncation Point k_L for the Standardized, Left-Truncated Normal Distribution

Clearly, monthly demand for this item cannot be less than 0 – so that this demand pattern would be most appropriately modeled as that of a left-truncated normal distribution. Using the notation of Figure 1: $\mu = 20$, $\sigma = 10$, and $x_L = 0$. In terms of the standard normal distribution, the truncation point k_L can be calculated using Equation 5

$$k_L = \frac{x_L - \mu}{\sigma} = \frac{0 - 20}{10} = -2$$

Finally, to develop the standardized, left-truncated normal distribution for this demand pattern, use the definitional relation $t = z - k_L$. In this case, $t = z + 2$.

The inventory manager's interest in a monthly demand of greater than 30 units ($P(x \geq 30)$) , then, has a standard normal equivalent of $P(z \geq 1)$ and a standardized, left-truncated normal equivalent of $P(t \geq 3)$.

$$P(x \geq 30) = P\left(z \geq \frac{30 - 20}{10}\right) = P(z \geq 1) = P(t \geq 1 + 2) = P(t \geq 3)$$

Using Table 1, the cumulative distribution function associated with a standardized t value of 3 and truncation point $k_L = -2$ can be found to be $F_{SLTN}(t = 3) = 0.8377$.

Thus, the probability that the level of demand will exceed 30 units is given by

$$P(x \geq 30) = P(t \geq 3) = 1 - F(t = 3)_{k_L = -2} = 1 - 0.8377 = 0.1633$$

In other words, there is a 16.33% chance that the monthly demand will exceed 30 units. Contrast this with the value of 15.87% that would be obtained using tables of the standard (non-truncated) normal distribution.

Table 2. Continuing the inventory management example above, suppose that the manager desires to insure that enough stock is on hand to satisfy 95% of the likely demand – i.e., to achieve a 95% service level. What stock level would be appropriate?

Using Table 2, for a truncation point of k_L = -2, to achieve $F_{SLTN}(t)$ = 0.95 requires a standardized t value of t = 3.6560. This, then, implies a standard normal value z of

$$z = t + k = 3.6560 - 2 = 1.6560$$

and a "real-world" stock level of

$$x = \mu + z\sigma = 20 + 1.6560(10) = 36.56$$

In other words, the manager should plan for an inventory level of 36.56 units in order to be able to meet the 95th percentile of the demands. Again, this value can be contrasted with the one of 36.45 units that would be obtained using non-truncated distributions.

The difference in these values, although small for this example, illustrates the fact that the Table 2 values account for the fact that the mean of the truncated distribution is actually 20.552 units (slightly greater than the non-truncated distribution) and its standard deviation is actually 9.415 units (slightly less than the non-truncated).

Table 3. The inventory manager now needs to develop an inventory plan for another item based upon a large ($n \geq 30$) sample of monthly demands – which have a sample mean of 20 units and a sample standard deviation of 10.4 units. Based upon these sample data, what would be the appropriate point of truncation to assume?

Using Equation 21, the coefficient of variation can be estimated to be $c = 0.52$. From Table 3, this implies a truncation point of $k_L = -1.60$.

Reference Visualizations for the Standardized, Left-Truncated Normal Distribution

This section presents visualizations of the standardized, left-truncated normal distribution for reference in working with this family of distributions. Specifically – for lower truncations points k_L of -3, -2, -1, 0, 1, 2, and 3 – this paper provides figures that graphically represent the probability density function $f_{SLTN}(t)$ and the cumulative probability distribution $F_{SLTN}(t)$ for the appropriate range of standard t values. In addition, the mean, standard deviation, and coefficient of variation are summarized for each of these figures – based upon the values determined in Table 2.

In Figures 8 through 14, the dashed line is that of the probability density function $f_{SLTN}(t)$ and the solid line is for the cumulative distribution function $F_{SLTN}(t)$.

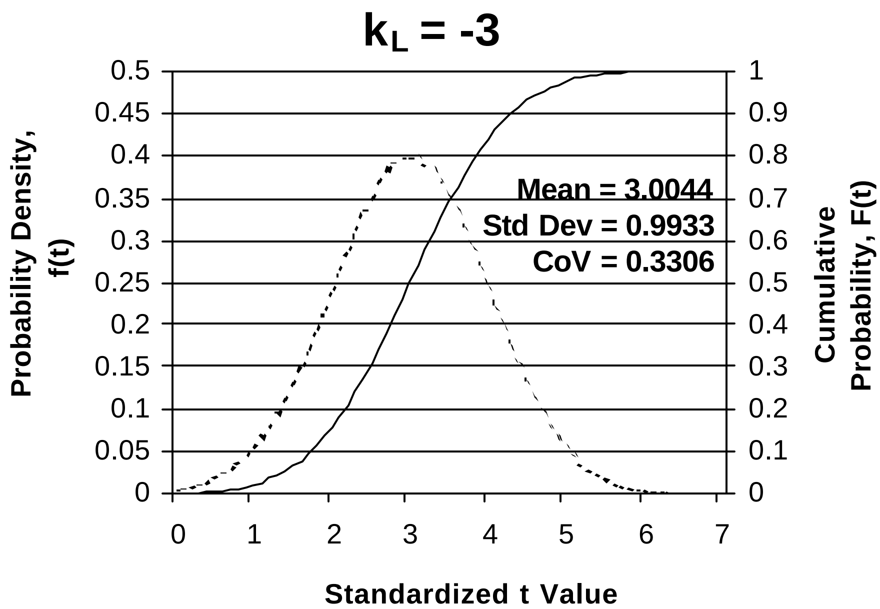

Figure 8. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation k_L = -3

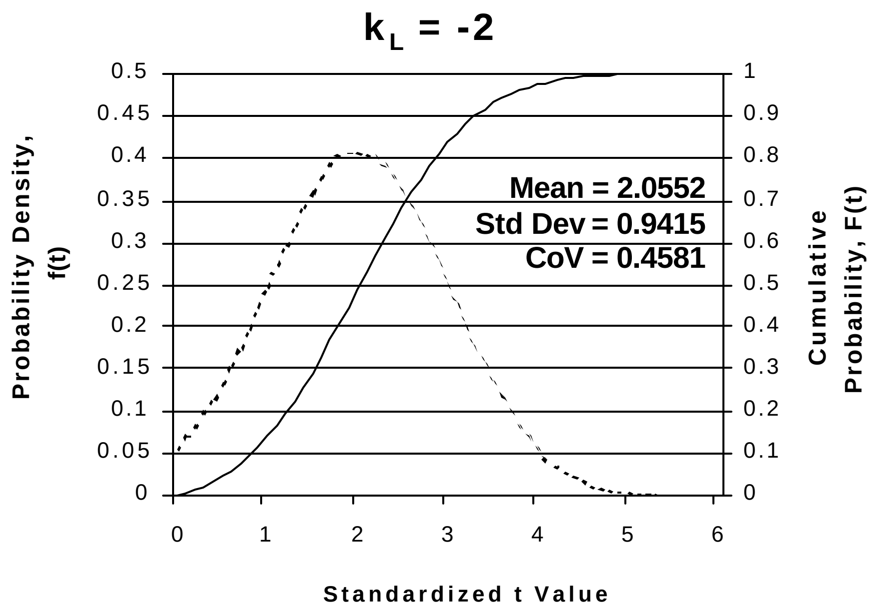

Figure 9. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation k_L = -2

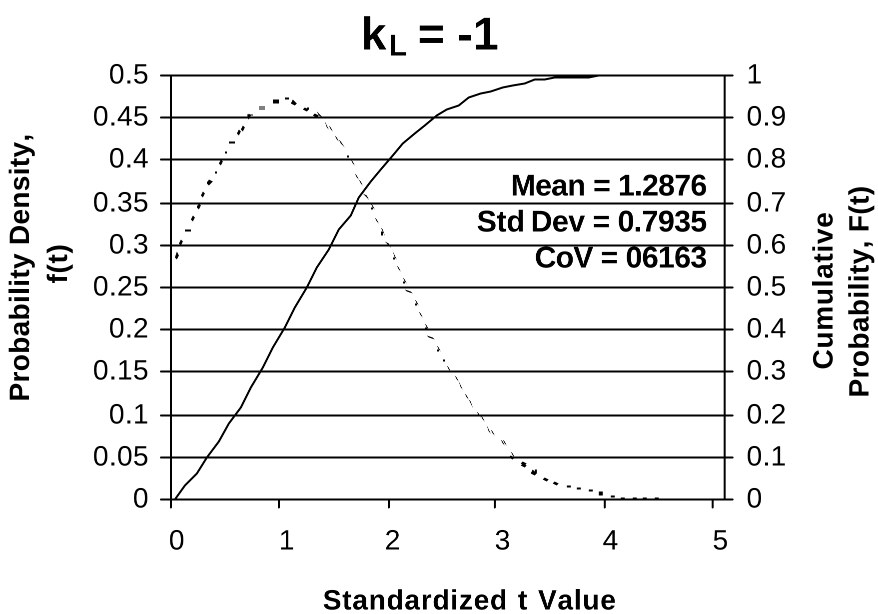

Figure 10. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation k_L = -1

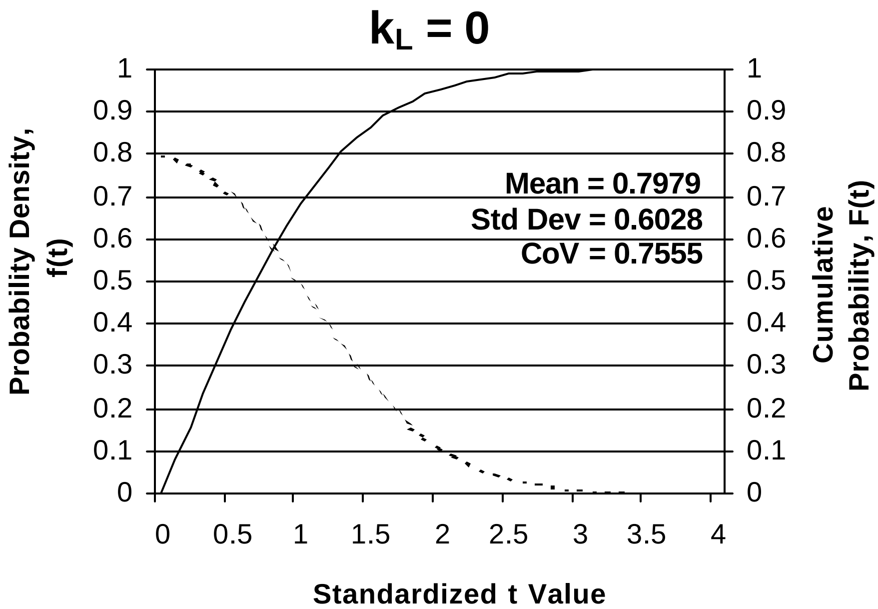

Figure 11. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation $k_L = 0$

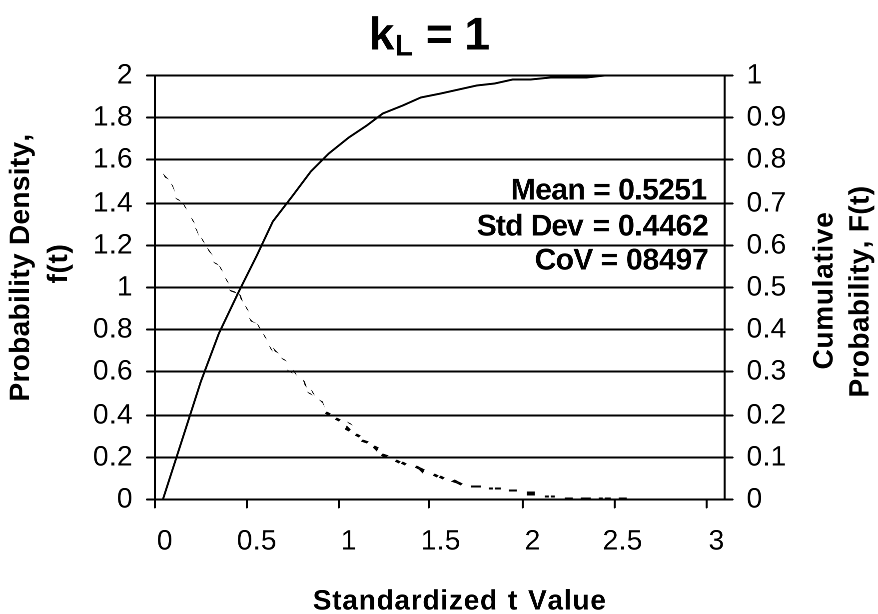

Figure 12. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation $k_L = 1$

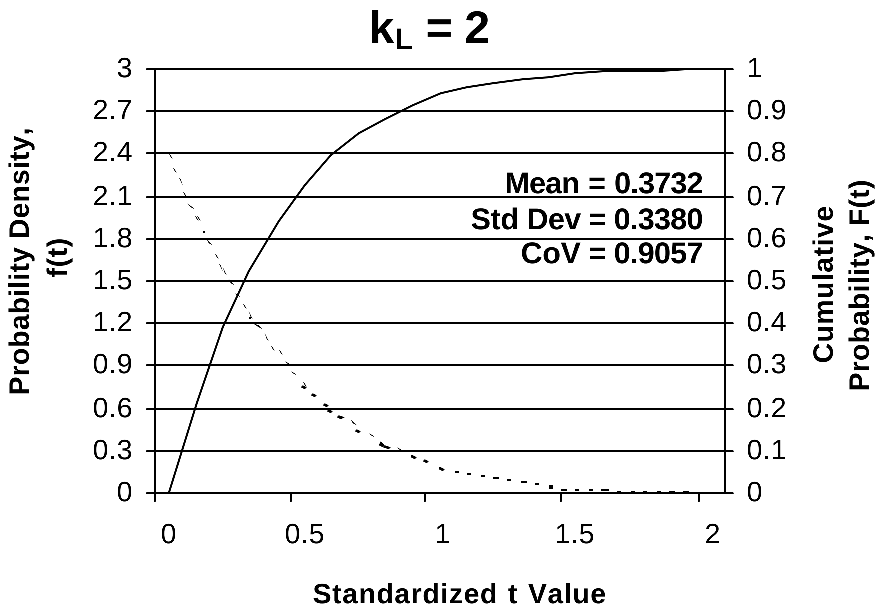

Figure 13. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation k_L = 2

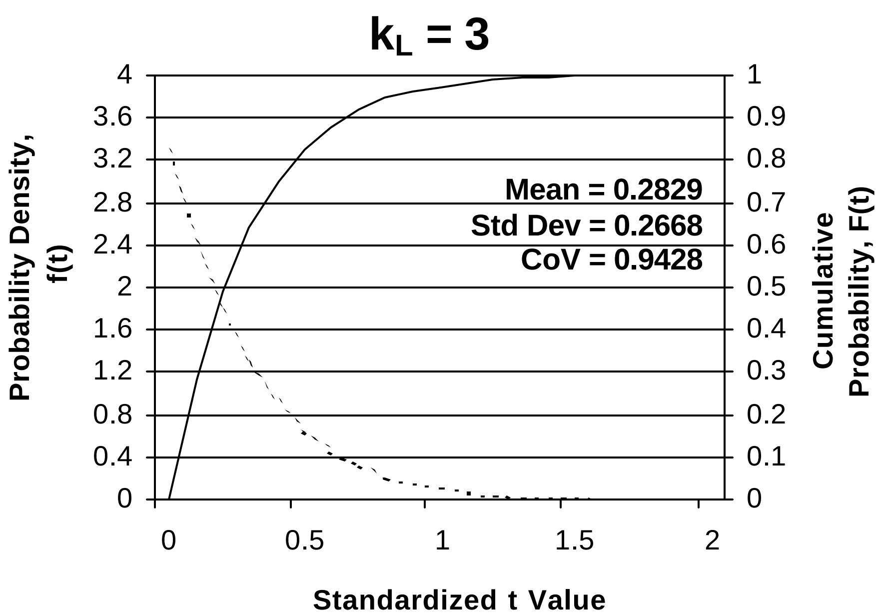

Figure 14. Visualization of the Standardized, Left-Truncated Normal Distribution with Point of Truncation k_L = 3

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