THE CUMULATIVE DISTRIBUTION FUNCTION OF THE RIGHT-TRUNCATED NORMAL DISTRIBUTION

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Definition of the Standardized, Right-Truncated Normal Distribution

Consider a normally-distributed random variable x with a probability density function f(x) specified as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty \qquad (1)$$

If the values of x above some value x_R cannot be observed – due to censoring or truncation – then, as shown in Figure 1, the resulting distribution is a right-truncated normal distribution with probability density function $f_{RTN}(x)$ given by

$$f_{RTN}(x) = \begin{cases} \dfrac{f(x)}{\int_{-\infty}^{x_R} f(x)dx}, & -\infty \le x \le x_R \\ 0, & x_R \le x \le \infty \end{cases} \qquad (2)$$

where f(x) is as defined in Equation 1.

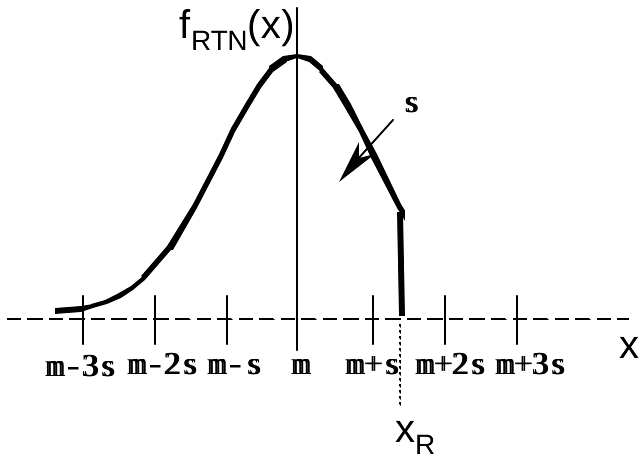

Figure 1. Right-Truncated Normal Distribution (in Terms of the Original Population Parameters)

For purposes of generality, Equation 2 can be re-stated in terms of the standard normal distribution (denoted f(z)) where

$$z = \frac{x - \mu}{\sigma} \tag{3}$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty \leq z \leq \infty \tag{4}$$

In terms of this standard normal distribution, the point of truncation x_R will be denoted k_R as given by

$$k_R = \frac{x_R - \mu}{\sigma} \tag{5}$$

Reformulating the right-truncated normal distribution of Equation 2 in terms of the standard normal distribution, the following can be found:

$$f_{RTN}(z) = \begin{cases} \dfrac{f(z)}{\int_{-\infty}^{k_R} f(z)dz}, & -\infty \le z \le k_R \\ 0, & k_R \le z \le \infty \end{cases} \qquad (6)$$

This is illustrated in Figure 2.

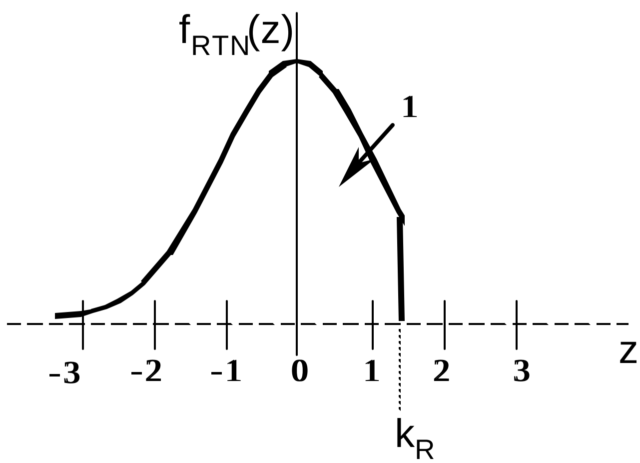

Figure 2. Right-Truncated Normal Distribution (in terms of the Standard Normal Distribution)

To define what this paper terms as a "standardized, right-truncated normal distribution," a standardizing variable $t = z - k_R$ is introduced, which has the effect of defining the point of truncation as $t = 0$. The standardized, right-truncated normal distribution $f_{SRTN}(t)$ is, thus, given by

$$f_{SRTN}(t) = \begin{cases} \dfrac{f(t + k_R)}{\int_{-\infty}^{k_R} f(z)dz}, & t \leq 0 \\ 0, & t \geq 0 \end{cases} \tag{7}$$

and is illustrated in Figure 3.

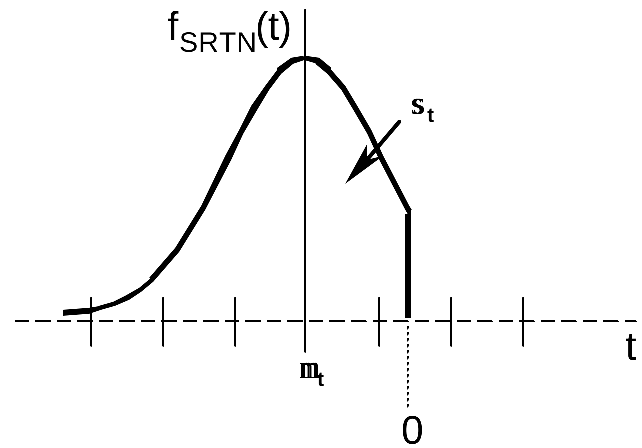

Figure 3. Standardized, Right-Truncated Normal Distribution

This paper, next, develops formulas for the two parameters of the standardized, right-truncated normal distribution: the mean (μ_t) and the standard deviation (σ_t) – for a given point of truncation k_R.

Parameters of the Standardized, Right-Truncated Normal Distribution

Consider the mean (μ_t) of the standardized, right-truncated normal distribution – for a given point of truncation k_R, where f(t) is as defined in Equation 7.

$$\mu_t = E(t) = \int_{-\infty}^{0} t f(t) dt \tag{8}$$

With t = z – k_R, it follows that dt = dz, z = k_R for t = 0, and z = -∞ for t = -∞. Using this, Equation 8 becomes

$$\begin{aligned} \mu_t &= \frac{1}{\int_{-\infty}^{k_R} f(z) dz} \int_{-\infty}^{k_R} (z - k_R) f(z) dz \\ &= \frac{1}{\int_{-\infty}^{k_R} f(z) dz} \left[\int_{-\infty}^{k_R} z f(z) dz - k_R \int_{-\infty}^{k_R} f(z) dz \right] \end{aligned} \tag{9}$$

Consider the first term within the brackets in Equation 9. To evaluate this term, one can make use of the identity [Bey87]

$$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \tag{10}$$

Using Equations 4 and 10, it can be shown that

$$\int_{-\infty}^{k_R} z f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k_R} z e^{-\frac{1}{2}z^2} dz = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{k_R/\sqrt{2}} x e^{-x^2} dx \tag{11}$$
$$= -\frac{1}{\sqrt{2\pi}} e^{-x^2} \Big|_{-\infty}^{k_R/\sqrt{2}} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k_R{}^2} = -f(k_R)$$

Defining F(k) as the cumulative distribution function for the standard normal variable value k

$$F(k) = \int_{-\infty}^{k} f(z) dz \tag{12}$$

and using Equations 11 and 12, Equation 9 can be re-stated in its final form as

$$\mu_t = -\frac{1}{F(k_R)} \left[f(k_R) + k_R F(k_R) \right] \tag{13}$$

From Equation 13, it is clear that the mean of the standardized, right-truncated normal distribution is uniquely determined by and solely dependent upon the point of truncation k_R.

Consider now the standard deviation (σ_t) of the standardized, right-truncated normal distribution. Given that [Hin90]

$$\sigma_t^2 = E(t^2) - \mu_t^2 \tag{14}$$

It is only necessary to find $E(t^2)$.

$$E(t^2) = \int_{-\infty}^{0} t^2 f(t)dt = \frac{1}{F(k_R)} \int_{-\infty}^{k_R} (z - k_R)^2 f(z)dz$$
$$= \frac{1}{F(k_R)} \left[\int_{-\infty}^{k_R} z^2 f(z)dz - 2k_R \int_{-\infty}^{k_R} z f(z)dz + k_R^2 \int_{-\infty}^{k_R} f(z)dz \right] \quad (15)$$

The last two terms within the brackets in Equation 15 have already been encountered in Equations 11 and 12, respectively. Consider, now, the first term – which, via integration by parts, can be found to be equivalent to

$$\int_{-\infty}^{k_R} z^2 f(z)dz = -z f(z)\Big|_{-\infty}^{k_R} + \int_{-\infty}^{k_R} f(z)dz = -k_R f(k_R) + F(k_R) \quad (16)$$

Utilizing Equations 11, 12, and 16, Equation 15 can be re-stated as

$$E(t^2) = \frac{1}{F(k_R)} \left[- k_R f(k_R) + F(k_R) + 2k_R f(k_R) + k_R^2 F(k_R) \right]$$
$$= \frac{1}{F(k_R)} \left[(1 + k_R^2) F(k_R) + k_R f(k_R) \right] \quad (17)$$

Equation 17 can be used along with Equation 13 to calculate σ_t as shown in Equation 14. Once again, it is worth noting that the point of truncation k_R again uniquely and solely determines the standard deviation of the standardized, right-truncated normal distribution.

Finally, it is convenient to define a coefficient of variation c – which, again, exists uniquely for a particular k_R

$$c = \frac{\sigma_t}{\mu_t} \tag{4-18}$$

Development of Tables of the Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution

Given the formulation of the probability density function $f_{SRTN}(t)$ of the standardized, right-truncated normal distribution in Equation 7, its cumulative distribution function $F_{SRTN}(t)$ can be stated as

$$F_{SRTN}(t) = \begin{cases} \dfrac{F(t + k_R)}{F(k_R)}, & t \le 0 \\ 1, & t \ge 0 \end{cases} \qquad (19)$$

where F(z) is as defined in Equation 12 and is the cumulative probability associated with a standard normal variate of value z.

Thus, given a point of truncation k_R, one can readily calculate the value of the cumulative distribution function of the standardized, right-truncated normal distribution at any standardized value of $t \le 0$ through the use of Equation 19 and tables of the cumulative distribution function of the standard normal distribution F(z).

With the wide availability of desktop micro-computers and their associated software, alternate methods of evaluating Equation 19 at specific values of t include the use of “standard” spreadsheet functions – e.g., NormSDist(Z) in Microsoft Excel – or formulations such as those provided in [Abr72]. In developing the tables presented below, the results of utilizing the Excel functions were compared with [Abramowitz, 1972]’s smallest error formulation: ($|\varepsilon(x)| < 7.5 \times 10^{-8}$)

$$F(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) \qquad (4\text{-}20)$$

where

$$t = \frac{1}{1 + px}$$

$$p = 0.231619,\ b_1 = 0.319381530,\ b_2 = -0.356563782,$$

$$b_3 = 1.781477937,\ b_4 = -1.821255978,\ b_5 = 1.330274429$$

In general, it was found that the results of the Excel-based calculations and those based upon Equation 20 were close enough that either could be used for purposes of developing the tables presented in this paper. Specifically, the values of the cumulative distribution function calculated using the Excel worksheet functions agreed with those tabulated in [Abr72] to at least six decimal places. Even after computations involved with the most heavily-truncated case presented in this paper (i.e., k_R = -3.0), Excel’s results agreed with those obtained using the tabulated values in [Abr72] to at least four decimal places. Based upon its ease of implementation, the Excel-based approach was used – with the results/values being shown to four decimal places.

Using the approach outlined above, a table of the cumulative distribution function $F_{SRTN}(t)$ for the standardized, right-truncated normal distribution as a function of the truncation point k_R has been developed. Table 1, entitled "Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution," is presented in five, overlapping segments divided by point of truncation – with the range and granularity of standardized t's over which $F_{SRTN}(t)$ is evaluated tailored to best illustrate the properties of the cumulative distribution function.

Using Table 1 requires only knowledge of the point of truncation k_R (in normalized terms) and, for a given point of truncation, can be utilized in a fashion comparable to other cumulative distribution tables for the (non-truncated) standard normal distribution. Detailed examples of the use of Table 1 will be presented at the end of this paper.

Figures 4 through 6 illustrate the cumulative distribution function of the standardized, right-truncated normal distribution function for seven values of the point of truncation (k_R = -3, -2, -1, 0, 1, 2, and 3) – over ranges of standardize t's of 0 to -6, 0 to -3, and 0 to -1.5, respectively.

The "inverse" of the problem of identifying the value of the cumulative distribution $F_{SRTN}(t)$ at a standardized value t – for a given point of truncation k_R – is that of identifying the standardized value t at which the cumulative distribution function $F_{SRTN}(t)$ assumes some value. While Table 1 can, in theory, be used for such purposes – subject to the granularity of the tabulated $F_{SRTN}(t)$ values – this problem has been solved computationally through the use of a dichotomous line search algorithm, and the results are presented in Table 2 over the range $F_{SRTN}(t) = 0.005$ to $F_{SRTN}(t) = 0.995$, with varying granularity. Table 2 is presented in two segments – for negative and positive values, respectively, of the point of truncation k_R over the range -3.0(0.2)3.0.

Table 1. Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution

t/k_R	3.0	2.8	2.6	2.4	2.2	2.0	1.8	1.6	1.4	1.2	1.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.2	0.9988	0.9979	0.9964	0.9942	0.9910	0.9865	0.9804	0.9725	0.9627	0.9507	0.9368
-0.4	0.9967	0.9943	0.9907	0.9853	0.9777	0.9672	0.9535	0.9362	0.9153	0.8906	0.8626
-0.6	0.9931	0.9886	0.9818	0.9720	0.9585	0.9406	0.9179	0.8901	0.8574	0.8201	0.7790
-0.8	0.9874	0.9798	0.9686	0.9530	0.9322	0.9055	0.8727	0.8338	0.7895	0.7406	0.6885
-1.0	0.9786	0.9665	0.9496	0.9268	0.8974	0.8609	0.8175	0.7678	0.7130	0.6546	0.5943
-1.2	0.9654	0.9476	0.9235	0.8922	0.8532	0.8065	0.7528	0.6934	0.6301	0.5650	0.5001
-1.4	0.9465	0.9216	0.8891	0.8483	0.7993	0.7426	0.6798	0.6128	0.5439	0.4755	0.4096
-1.6	0.9205	0.8872	0.8453	0.7947	0.7360	0.6707	0.6008	0.5290	0.4577	0.3894	0.3260
-1.8	0.8861	0.8435	0.7918	0.7317	0.6647	0.5927	0.5186	0.4451	0.3748	0.3099	0.2518
-2.0	0.8425	0.7902	0.7291	0.6608	0.5874	0.5116	0.4364	0.3646	0.2983	0.2394	0.1886
-2.2	0.7892	0.7276	0.6585	0.5840	0.5070	0.4305	0.3574	0.2902	0.2305	0.1793	0.1368
-2.4	0.7267	0.6571	0.5820	0.5041	0.4267	0.3526	0.2845	0.2241	0.1726	0.1300	0.0960
-2.6	0.6563	0.5807	0.5023	0.4242	0.3494	0.2806	0.2198	0.1679	0.1252	0.0913	0.0651
-2.8	0.5800	0.5013	0.4227	0.3474	0.2781	0.2168	0.1646	0.1217	0.0879	0.0619	0.0427
-3.0	0.5007	0.4218	0.3462	0.2765	0.2148	0.1623	0.1194	0.0854	0.0596	0.0406	0.0270
-3.2	0.4213	0.3455	0.2755	0.2136	0.1609	0.1177	0.0838	0.0580	0.0391	0.0257	0.0165
-3.4	0.3450	0.2750	0.2128	0.1600	0.1167	0.0826	0.0568	0.0380	0.0247	0.0157	0.0097
-3.6	0.2746	0.2124	0.1594	0.1160	0.0819	0.0561	0.0373	0.0241	0.0151	0.0093	0.0055
-3.8	0.2121	0.1591	0.1156	0.0814	0.0556	0.0368	0.0236	0.0147	0.0089	0.0053	0.0030
-4.0	0.1589	0.1154	0.0811	0.0553	0.0364	0.0233	0.0144	0.0087	0.0051	0.0029	0.0016
-4.2	0.1152	0.0810	0.0551	0.0362	0.0231	0.0142	0.0085	0.0049	0.0028	0.0015	0.0008
-4.4	0.0809	0.0549	0.0361	0.0229	0.0141	0.0084	0.0048	0.0027	0.0015	0.0008	0.0004
-4.6	0.0549	0.0360	0.0229	0.0140	0.0083	0.0048	0.0027	0.0014	0.0007	0.0004	0.0002
-4.8	0.0360	0.0228	0.0140	0.0083	0.0047	0.0026	0.0014	0.0007	0.0004	0.0002	0.0001
-5.0	0.0228	0.0139	0.0082	0.0047	0.0026	0.0014	0.0007	0.0004	0.0002	0.0001	0.0000
-5.2	0.0139	0.0082	0.0047	0.0026	0.0014	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000
-5.4	0.0082	0.0047	0.0026	0.0014	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
-5.6	0.0047	0.0026	0.0014	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
-5.8	0.0026	0.0014	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
-6.0	0.0014	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-6.2	0.0007	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-6.4	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-6.6	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-6.8	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-7.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1. (continued)

t/k_R	2.0	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2	0.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.1	0.9939	0.9910	0.9873	0.9825	0.9767	0.9698	0.9618	0.9528	0.9428	0.9319	0.9203
-0.2	0.9865	0.9804	0.9725	0.9627	0.9507	0.9368	0.9208	0.9031	0.8838	0.8632	0.8415
-0.3	0.9777	0.9680	0.9556	0.9403	0.9220	0.9010	0.8773	0.8514	0.8236	0.7944	0.7642
-0.4	0.9672	0.9535	0.9362	0.9153	0.8906	0.8626	0.8316	0.7982	0.7629	0.7263	0.6892
-0.5	0.9549	0.9369	0.9144	0.8876	0.8566	0.8219	0.7840	0.7438	0.7021	0.6596	0.6171
-0.6	0.9406	0.9179	0.8901	0.8574	0.8201	0.7790	0.7350	0.6889	0.6419	0.5949	0.5485
-0.7	0.9242	0.8965	0.8632	0.8246	0.7814	0.7344	0.6849	0.6341	0.5830	0.5326	0.4839
-0.8	0.9055	0.8727	0.8338	0.7895	0.7406	0.6885	0.6344	0.5797	0.5257	0.4735	0.4237
-0.9	0.8845	0.8463	0.8020	0.7522	0.6983	0.6416	0.5839	0.5265	0.4707	0.4177	0.3681
-1.0	0.8609	0.8175	0.7678	0.7130	0.6546	0.5943	0.5338	0.4748	0.4184	0.3657	0.3173
-1.1	0.8349	0.7863	0.7316	0.6722	0.6100	0.5469	0.4848	0.4251	0.3692	0.3178	0.2713
-1.2	0.8065	0.7528	0.6934	0.6301	0.5650	0.5001	0.4372	0.3779	0.3232	0.2739	0.2301
-1.3	0.7757	0.7172	0.6537	0.5873	0.5200	0.4541	0.3915	0.3334	0.2808	0.2342	0.1936
-1.4	0.7426	0.6798	0.6128	0.5439	0.4755	0.4096	0.3480	0.2919	0.2421	0.1986	0.1615
-1.5	0.7076	0.6409	0.5711	0.5006	0.4318	0.3667	0.3070	0.2536	0.2070	0.1671	0.1336
-1.6	0.6707	0.6008	0.5290	0.4577	0.3894	0.3260	0.2688	0.2186	0.1756	0.1394	0.1096
-1.7	0.6323	0.5599	0.4869	0.4157	0.3487	0.2876	0.2335	0.1869	0.1477	0.1153	0.0891
-1.8	0.5927	0.5186	0.4451	0.3748	0.3099	0.2518	0.2013	0.1586	0.1232	0.0946	0.0719
-1.9	0.5524	0.4773	0.4042	0.3356	0.2734	0.2188	0.1721	0.1334	0.1019	0.0769	0.0574
-2.0	0.5116	0.4364	0.3646	0.2983	0.2394	0.1886	0.1460	0.1113	0.0836	0.0620	0.0455
-2.1	0.4709	0.3963	0.3264	0.2632	0.2080	0.1612	0.1228	0.0921	0.0680	0.0496	0.0357
-2.2	0.4305	0.3574	0.2902	0.2305	0.1793	0.1368	0.1025	0.0755	0.0548	0.0393	0.0278
-2.3	0.3910	0.3200	0.2560	0.2002	0.1533	0.1151	0.0848	0.0614	0.0438	0.0308	0.0214
-2.4	0.3526	0.2845	0.2241	0.1726	0.1300	0.0960	0.0695	0.0495	0.0347	0.0240	0.0164
-2.5	0.3157	0.2510	0.1947	0.1476	0.1094	0.0794	0.0565	0.0396	0.0273	0.0185	0.0124
-2.6	0.2806	0.2198	0.1679	0.1252	0.0913	0.0651	0.0456	0.0313	0.0212	0.0142	0.0093
-2.7	0.2476	0.1909	0.1435	0.1053	0.0755	0.0530	0.0364	0.0246	0.0164	0.0107	0.0069
-2.8	0.2168	0.1646	0.1217	0.0879	0.0619	0.0427	0.0289	0.0192	0.0125	0.0080	0.0051
-2.9	0.1883	0.1407	0.1024	0.0727	0.0504	0.0341	0.0227	0.0148	0.0095	0.0060	0.0037
-3.0	0.1623	0.1194	0.0854	0.0596	0.0406	0.0270	0.0176	0.0113	0.0071	0.0044	0.0027
-3.1	0.1388	0.1004	0.0707	0.0485	0.0325	0.0212	0.0136	0.0086	0.0053	0.0032	0.0019
-3.2	0.1177	0.0838	0.0580	0.0391	0.0257	0.0165	0.0104	0.0064	0.0039	0.0023	0.0014
-3.3	0.0991	0.0693	0.0471	0.0312	0.0202	0.0127	0.0079	0.0048	0.0028	0.0017	0.0010
-3.4	0.0826	0.0568	0.0380	0.0247	0.0157	0.0097	0.0059	0.0035	0.0021	0.0012	0.0007
-3.5	0.0684	0.0462	0.0304	0.0194	0.0121	0.0074	0.0044	0.0026	0.0015	0.0008	0.0005
-3.6	0.0561	0.0373	0.0241	0.0151	0.0093	0.0055	0.0032	0.0019	0.0010	0.0006	0.0003
-3.7	0.0456	0.0298	0.0189	0.0117	0.0070	0.0041	0.0024	0.0013	0.0007	0.0004	0.0002
-3.8	0.0368	0.0236	0.0147	0.0089	0.0053	0.0030	0.0017	0.0009	0.0005	0.0003	0.0001
-3.9	0.0294	0.0185	0.0113	0.0068	0.0039	0.0022	0.0012	0.0007	0.0004	0.0002	0.0001
-4.0	0.0233	0.0144	0.0087	0.0051	0.0029	0.0016	0.0009	0.0005	0.0002	0.0001	0.0001

Table 1. (continued)

t/k_R	1.0	0.8	0.6	0.4	0.2	0.0	-0.2	-0.4	-0.6	-0.8	-1.0
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.05	0.9853	0.9813	0.9767	0.9716	0.9661	0.9601	0.9538	0.9471	0.9402	0.9330	0.9256
-0.10	0.9698	0.9618	0.9528	0.9428	0.9319	0.9203	0.9081	0.8954	0.8823	0.8688	0.8551
-0.15	0.9536	0.9416	0.9282	0.9135	0.8976	0.8808	0.8632	0.8450	0.8263	0.8074	0.7883
-0.20	0.9368	0.9208	0.9031	0.8838	0.8632	0.8415	0.8190	0.7959	0.7725	0.7489	0.7253
-0.25	0.9192	0.8994	0.8775	0.8538	0.8287	0.8026	0.7757	0.7483	0.7207	0.6932	0.6659
-0.30	0.9010	0.8773	0.8514	0.8236	0.7944	0.7642	0.7333	0.7022	0.6711	0.6404	0.6101
-0.35	0.8821	0.8547	0.8250	0.7933	0.7603	0.7263	0.6920	0.6577	0.6237	0.5904	0.5579
-0.40	0.8626	0.8316	0.7982	0.7629	0.7263	0.6892	0.6518	0.6148	0.5785	0.5432	0.5090
-0.45	0.8425	0.8080	0.7711	0.7324	0.6928	0.6527	0.6128	0.5736	0.5355	0.4987	0.4635
-0.50	0.8219	0.7840	0.7438	0.7021	0.6596	0.6171	0.5751	0.5342	0.4947	0.4569	0.4211
-0.55	0.8007	0.7596	0.7164	0.6719	0.6270	0.5823	0.5386	0.4964	0.4560	0.4178	0.3818
-0.60	0.7790	0.7350	0.6889	0.6419	0.5949	0.5485	0.5035	0.4604	0.4196	0.3812	0.3454
-0.65	0.7569	0.7100	0.6615	0.6123	0.5634	0.5157	0.4698	0.4262	0.3852	0.3471	0.3118
-0.70	0.7344	0.6849	0.6341	0.5830	0.5326	0.4839	0.4375	0.3937	0.3530	0.3153	0.2809
-0.75	0.7116	0.6597	0.6068	0.5541	0.5026	0.4533	0.4066	0.3630	0.3227	0.2859	0.2525
-0.80	0.6885	0.6344	0.5797	0.5257	0.4735	0.4237	0.3771	0.3339	0.2945	0.2587	0.2265
-0.85	0.6651	0.6091	0.5529	0.4979	0.4451	0.3953	0.3490	0.3066	0.2681	0.2335	0.2027
-0.90	0.6416	0.5839	0.5265	0.4707	0.4177	0.3681	0.3224	0.2809	0.2436	0.2104	0.1810
-0.95	0.6180	0.5588	0.5004	0.4442	0.3912	0.3421	0.2973	0.2569	0.2209	0.1891	0.1613
-1.00	0.5943	0.5338	0.4748	0.4184	0.3657	0.3173	0.2735	0.2344	0.1998	0.1696	0.1434
-1.05	0.5706	0.5092	0.4497	0.3934	0.3412	0.2937	0.2511	0.2134	0.1804	0.1518	0.1272
-1.10	0.5469	0.4848	0.4251	0.3692	0.3178	0.2713	0.2301	0.1939	0.1625	0.1355	0.1126
-1.15	0.5234	0.4608	0.4012	0.3458	0.2953	0.2501	0.2104	0.1758	0.1461	0.1208	0.0994
-1.20	0.5001	0.4372	0.3779	0.3232	0.2739	0.2301	0.1919	0.1590	0.1310	0.1074	0.0876
-1.25	0.4770	0.4141	0.3553	0.3016	0.2535	0.2113	0.1748	0.1436	0.1173	0.0953	0.0771
-1.30	0.4541	0.3915	0.3334	0.2808	0.2342	0.1936	0.1588	0.1293	0.1047	0.0843	0.0676
-1.35	0.4317	0.3694	0.3123	0.2610	0.2159	0.1770	0.1440	0.1163	0.0933	0.0745	0.0592
-1.40	0.4096	0.3480	0.2919	0.2421	0.1986	0.1615	0.1302	0.1043	0.0830	0.0656	0.0517
-1.45	0.3879	0.3272	0.2724	0.2241	0.1824	0.1471	0.1176	0.0933	0.0736	0.0577	0.0450
-1.50	0.3667	0.3070	0.2536	0.2070	0.1671	0.1336	0.1059	0.0833	0.0651	0.0506	0.0391
-1.55	0.3461	0.2875	0.2357	0.1908	0.1528	0.1211	0.0952	0.0743	0.0575	0.0443	0.0339
-1.60	0.3260	0.2688	0.2186	0.1756	0.1394	0.1096	0.0854	0.0660	0.0507	0.0387	0.0294
-1.65	0.3065	0.2508	0.2024	0.1612	0.1269	0.0989	0.0764	0.0586	0.0446	0.0337	0.0254
-1.70	0.2876	0.2335	0.1869	0.1477	0.1153	0.0891	0.0683	0.0518	0.0391	0.0293	0.0219
-1.75	0.2694	0.2170	0.1723	0.1350	0.1046	0.0801	0.0608	0.0458	0.0342	0.0254	0.0188
-1.80	0.2518	0.2013	0.1586	0.1232	0.0946	0.0719	0.0541	0.0403	0.0299	0.0220	0.0161
-1.85	0.2349	0.1863	0.1456	0.1122	0.0854	0.0643	0.0480	0.0355	0.0260	0.0190	0.0138
-1.90	0.2188	0.1721	0.1334	0.1019	0.0769	0.0574	0.0425	0.0311	0.0226	0.0164	0.0118
-1.95	0.2033	0.1587	0.1220	0.0924	0.0692	0.0512	0.0375	0.0272	0.0196	0.0141	0.0100
-2.00	0.1886	0.1460	0.1113	0.0836	0.0620	0.0455	0.0330	0.0238	0.0170	0.0121	0.0085
-2.05	0.1746	0.1340	0.1013	0.0755	0.0555	0.0404	0.0291	0.0207	0.0147	0.0103	0.0072
-2.10	0.1612	0.1228	0.0921	0.0680	0.0496	0.0357	0.0255	0.0180	0.0126	0.0088	0.0061
-2.15	0.1487	0.1123	0.0835	0.0611	0.0442	0.0316	0.0223	0.0156	0.0109	0.0075	0.0051
-2.20	0.1368	0.1025	0.0755	0.0548	0.0393	0.0278	0.0195	0.0135	0.0093	0.0064	0.0043
-2.25	0.1256	0.0933	0.0682	0.0491	0.0348	0.0244	0.0170	0.0117	0.0080	0.0054	0.0036
-2.30	0.1151	0.0848	0.0614	0.0438	0.0308	0.0214	0.0148	0.0101	0.0068	0.0046	0.0030
-2.35	0.1052	0.0769	0.0552	0.0390	0.0272	0.0188	0.0128	0.0086	0.0058	0.0039	0.0025
-2.40	0.0960	0.0695	0.0495	0.0347	0.0240	0.0164	0.0111	0.0074	0.0049	0.0032	0.0021
-2.45	0.0874	0.0628	0.0443	0.0308	0.0211	0.0143	0.0096	0.0063	0.0042	0.0027	0.0018
-2.50	0.0794	0.0565	0.0396	0.0273	0.0185	0.0124	0.0082	0.0054	0.0035	0.0023	0.0015

Table 1. (continued)

t/k_R	0.0	-0.2	-0.4	-0.6	-0.8	-1.0	-1.2	-1.4	-1.6	-1.8	-2.0
0.000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.025	0.9801	0.9768	0.9734	0.9699	0.9662	0.9623	0.9584	0.9545	0.9504	0.9463	0.9421
-0.050	0.9601	0.9538	0.9471	0.9402	0.9330	0.9256	0.9181	0.9105	0.9028	0.8950	0.8871
-0.075	0.9402	0.9309	0.9211	0.9110	0.9006	0.8899	0.8791	0.8681	0.8571	0.8460	0.8349
-0.100	0.9203	0.9081	0.8954	0.8823	0.8688	0.8551	0.8412	0.8273	0.8132	0.7992	0.7852
-0.125	0.9005	0.8856	0.8700	0.8541	0.8378	0.8212	0.8046	0.7879	0.7712	0.7546	0.7382
-0.150	0.8808	0.8632	0.8450	0.8263	0.8074	0.7883	0.7692	0.7500	0.7310	0.7122	0.6935
-0.175	0.8611	0.8410	0.8203	0.7992	0.7778	0.7563	0.7349	0.7136	0.6925	0.6717	0.6512
-0.200	0.8415	0.8190	0.7959	0.7725	0.7489	0.7253	0.7018	0.6786	0.6557	0.6332	0.6111
-0.225	0.8220	0.7972	0.7719	0.7463	0.7207	0.6951	0.6698	0.6449	0.6205	0.5965	0.5732
-0.250	0.8026	0.7757	0.7483	0.7207	0.6932	0.6659	0.6390	0.6126	0.5868	0.5617	0.5373
-0.275	0.7833	0.7544	0.7251	0.6957	0.6664	0.6376	0.6093	0.5816	0.5547	0.5286	0.5034
-0.300	0.7642	0.7333	0.7022	0.6711	0.6404	0.6101	0.5806	0.5518	0.5240	0.4972	0.4714
-0.325	0.7452	0.7125	0.6797	0.6472	0.6150	0.5836	0.5530	0.5233	0.4948	0.4674	0.4411
-0.350	0.7263	0.6920	0.6577	0.6237	0.5904	0.5579	0.5264	0.4960	0.4669	0.4391	0.4126
-0.375	0.7077	0.6718	0.6361	0.6008	0.5664	0.5330	0.5008	0.4699	0.4404	0.4123	0.3857
-0.400	0.6892	0.6518	0.6148	0.5785	0.5432	0.5090	0.4762	0.4449	0.4152	0.3870	0.3603
-0.425	0.6708	0.6322	0.5940	0.5567	0.5206	0.4858	0.4526	0.4210	0.3911	0.3629	0.3364
-0.450	0.6527	0.6128	0.5736	0.5355	0.4987	0.4635	0.4299	0.3982	0.3683	0.3402	0.3140
-0.475	0.6348	0.5938	0.5537	0.5148	0.4775	0.4419	0.4082	0.3764	0.3466	0.3188	0.2928
-0.500	0.6171	0.5751	0.5342	0.4947	0.4569	0.4211	0.3873	0.3556	0.3260	0.2985	0.2730
-0.525	0.5996	0.5567	0.5151	0.4751	0.4370	0.4011	0.3673	0.3358	0.3065	0.2793	0.2543
-0.550	0.5823	0.5386	0.4964	0.4560	0.4178	0.3818	0.3481	0.3169	0.2879	0.2612	0.2368
-0.575	0.5653	0.5209	0.4782	0.4375	0.3992	0.3632	0.3298	0.2988	0.2704	0.2442	0.2203
-0.600	0.5485	0.5035	0.4604	0.4196	0.3812	0.3454	0.3122	0.2817	0.2537	0.2282	0.2049
-0.625	0.5320	0.4865	0.4431	0.4021	0.3638	0.3283	0.2955	0.2654	0.2380	0.2130	0.1904
-0.650	0.5157	0.4698	0.4262	0.3852	0.3471	0.3118	0.2795	0.2499	0.2231	0.1988	0.1769
-0.675	0.4997	0.4535	0.4097	0.3688	0.3309	0.2960	0.2642	0.2352	0.2090	0.1854	0.1642
-0.700	0.4839	0.4375	0.3937	0.3530	0.3153	0.2809	0.2496	0.2212	0.1957	0.1728	0.1524
-0.725	0.4685	0.4218	0.3781	0.3376	0.3003	0.2664	0.2356	0.2079	0.1831	0.1610	0.1413
-0.750	0.4533	0.4066	0.3630	0.3227	0.2859	0.2525	0.2224	0.1954	0.1713	0.1499	0.1310
-0.775	0.4383	0.3916	0.3482	0.3083	0.2720	0.2392	0.2097	0.1835	0.1601	0.1395	0.1213
-0.800	0.4237	0.3771	0.3339	0.2945	0.2587	0.2265	0.1977	0.1722	0.1496	0.1297	0.1123
-0.825	0.4094	0.3629	0.3201	0.2810	0.2458	0.2143	0.1863	0.1615	0.1397	0.1206	0.1039
-0.850	0.3953	0.3490	0.3066	0.2681	0.2335	0.2027	0.1754	0.1514	0.1303	0.1120	0.0961
-0.875	0.3816	0.3356	0.2936	0.2556	0.2217	0.1916	0.1651	0.1418	0.1216	0.1040	0.0888
-0.900	0.3681	0.3224	0.2809	0.2436	0.2104	0.1810	0.1552	0.1328	0.1133	0.0965	0.0820
-0.925	0.3550	0.3097	0.2687	0.2320	0.1995	0.1709	0.1459	0.1243	0.1056	0.0895	0.0757
-0.950	0.3421	0.2973	0.2569	0.2209	0.1891	0.1613	0.1371	0.1162	0.0983	0.0829	0.0698
-0.975	0.3296	0.2852	0.2454	0.2101	0.1791	0.1521	0.1287	0.1087	0.0915	0.0768	0.0644
-1.000	0.3173	0.2735	0.2344	0.1998	0.1696	0.1434	0.1208	0.1015	0.0851	0.0711	0.0593
-1.025	0.3054	0.2621	0.2237	0.1899	0.1605	0.1351	0.1133	0.0948	0.0791	0.0658	0.0546
-1.050	0.2937	0.2511	0.2134	0.1804	0.1518	0.1272	0.1062	0.0884	0.0734	0.0608	0.0503
-1.075	0.2824	0.2404	0.2035	0.1713	0.1435	0.1197	0.0995	0.0825	0.0682	0.0562	0.0463
-1.100	0.2713	0.2301	0.1939	0.1625	0.1355	0.1126	0.0932	0.0769	0.0633	0.0519	0.0425
-1.125	0.2606	0.2201	0.1847	0.1541	0.1280	0.1058	0.0872	0.0716	0.0587	0.0479	0.0391
-1.150	0.2501	0.2104	0.1758	0.1461	0.1208	0.0994	0.0816	0.0667	0.0544	0.0442	0.0359
-1.175	0.2400	0.2010	0.1672	0.1384	0.1139	0.0934	0.0763	0.0621	0.0504	0.0408	0.0329
-1.200	0.2301	0.1919	0.1590	0.1310	0.1074	0.0876	0.0712	0.0577	0.0466	0.0376	0.0302
-1.225	0.2206	0.1832	0.1511	0.1240	0.1012	0.0822	0.0665	0.0536	0.0431	0.0346	0.0277
-1.250	0.2113	0.1748	0.1436	0.1173	0.0953	0.0771	0.0621	0.0498	0.0399	0.0318	0.0254

Table 1. (continued)

t/k_R	-1.0	-1.2	-1.4	-1.6	-1.8	-2.0	-2.2	-2.4	-2.6	-2.8	-3.0
0.000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.015	0.9773	0.9749	0.9725	0.9700	0.9675	0.9649	0.9624	0.9598	0.9571	0.9545	0.9518
-0.030	0.9549	0.9503	0.9455	0.9407	0.9358	0.9309	0.9259	0.9209	0.9159	0.9109	0.9058
-0.045	0.9329	0.9261	0.9192	0.9121	0.9051	0.8979	0.8907	0.8835	0.8763	0.8691	0.8619
-0.060	0.9112	0.9024	0.8934	0.8843	0.8751	0.8659	0.8567	0.8474	0.8382	0.8290	0.8198
-0.075	0.8899	0.8791	0.8681	0.8571	0.8460	0.8349	0.8238	0.8127	0.8016	0.7906	0.7797
-0.090	0.8689	0.8562	0.8434	0.8306	0.8177	0.8048	0.7919	0.7792	0.7665	0.7539	0.7414
-0.105	0.8483	0.8338	0.8193	0.8047	0.7901	0.7756	0.7612	0.7469	0.7327	0.7187	0.7048
-0.120	0.8279	0.8118	0.7957	0.7795	0.7634	0.7474	0.7315	0.7158	0.7003	0.6850	0.6699
-0.135	0.8080	0.7903	0.7726	0.7549	0.7374	0.7200	0.7028	0.6859	0.6692	0.6527	0.6366
-0.150	0.7883	0.7692	0.7500	0.7310	0.7122	0.6935	0.6751	0.6570	0.6393	0.6218	0.6048
-0.165	0.7690	0.7485	0.7280	0.7077	0.6876	0.6679	0.6484	0.6293	0.6106	0.5923	0.5744
-0.180	0.7501	0.7282	0.7065	0.6850	0.6638	0.6430	0.6226	0.6026	0.5831	0.5641	0.5455
-0.195	0.7314	0.7083	0.6855	0.6629	0.6407	0.6190	0.5977	0.5770	0.5567	0.5371	0.5180
-0.210	0.7131	0.6889	0.6649	0.6414	0.6183	0.5957	0.5737	0.5523	0.5314	0.5112	0.4917
-0.225	0.6951	0.6698	0.6449	0.6205	0.5965	0.5732	0.5505	0.5285	0.5072	0.4866	0.4666
-0.240	0.6775	0.6512	0.6254	0.6001	0.5754	0.5514	0.5282	0.5057	0.4839	0.4630	0.4428
-0.255	0.6602	0.6330	0.6063	0.5803	0.5549	0.5304	0.5067	0.4837	0.4617	0.4404	0.4200
-0.270	0.6432	0.6151	0.5877	0.5610	0.5351	0.5101	0.4859	0.4627	0.4403	0.4189	0.3984
-0.285	0.6265	0.5977	0.5695	0.5422	0.5158	0.4904	0.4659	0.4424	0.4199	0.3983	0.3778
-0.300	0.6101	0.5806	0.5518	0.5240	0.4972	0.4714	0.4466	0.4229	0.4003	0.3787	0.3581
-0.315	0.5941	0.5639	0.5346	0.5063	0.4791	0.4530	0.4281	0.4042	0.3816	0.3600	0.3395
-0.330	0.5784	0.5476	0.5178	0.4891	0.4616	0.4353	0.4102	0.3863	0.3636	0.3421	0.3217
-0.345	0.5629	0.5316	0.5014	0.4724	0.4446	0.4182	0.3930	0.3691	0.3464	0.3250	0.3048
-0.360	0.5478	0.5160	0.4855	0.4562	0.4282	0.4016	0.3764	0.3526	0.3300	0.3087	0.2887
-0.375	0.5330	0.5008	0.4699	0.4404	0.4123	0.3857	0.3605	0.3367	0.3143	0.2932	0.2734
-0.390	0.5185	0.4859	0.4548	0.4251	0.3969	0.3703	0.3452	0.3215	0.2993	0.2784	0.2589
-0.405	0.5043	0.4714	0.4401	0.4103	0.3820	0.3554	0.3304	0.3069	0.2849	0.2643	0.2451
-0.420	0.4904	0.4573	0.4257	0.3958	0.3676	0.3411	0.3162	0.2929	0.2712	0.2509	0.2320
-0.435	0.4768	0.4434	0.4118	0.3819	0.3537	0.3273	0.3026	0.2795	0.2580	0.2381	0.2195
-0.450	0.4635	0.4299	0.3982	0.3683	0.3402	0.3140	0.2895	0.2667	0.2455	0.2258	0.2077
-0.465	0.4504	0.4168	0.3850	0.3551	0.3272	0.3011	0.2769	0.2544	0.2335	0.2142	0.1964
-0.480	0.4377	0.4039	0.3722	0.3424	0.3146	0.2888	0.2648	0.2426	0.2221	0.2032	0.1857
-0.495	0.4252	0.3914	0.3597	0.3300	0.3024	0.2768	0.2531	0.2313	0.2111	0.1926	0.1756
-0.510	0.4130	0.3792	0.3475	0.3181	0.2907	0.2653	0.2420	0.2205	0.2007	0.1826	0.1660
-0.525	0.4011	0.3673	0.3358	0.3065	0.2793	0.2543	0.2312	0.2101	0.1907	0.1730	0.1569
-0.540	0.3894	0.3557	0.3243	0.2952	0.2683	0.2436	0.2210	0.2002	0.1812	0.1640	0.1482
-0.555	0.3780	0.3444	0.3132	0.2843	0.2578	0.2334	0.2111	0.1907	0.1722	0.1553	0.1400
-0.570	0.3669	0.3334	0.3024	0.2738	0.2475	0.2235	0.2016	0.1816	0.1635	0.1471	0.1322
-0.585	0.3560	0.3227	0.2919	0.2636	0.2377	0.2140	0.1925	0.1730	0.1553	0.1393	0.1249
-0.600	0.3454	0.3122	0.2817	0.2537	0.2282	0.2049	0.1838	0.1647	0.1474	0.1319	0.1179
-0.615	0.3350	0.3021	0.2718	0.2442	0.2190	0.1961	0.1754	0.1568	0.1399	0.1248	0.1113
-0.630	0.3249	0.2922	0.2622	0.2349	0.2101	0.1877	0.1674	0.1492	0.1328	0.1181	0.1050
-0.645	0.3151	0.2826	0.2529	0.2260	0.2016	0.1795	0.1597	0.1419	0.1260	0.1118	0.0991
-0.660	0.3054	0.2732	0.2439	0.2173	0.1933	0.1717	0.1524	0.1350	0.1195	0.1057	0.0934
-0.675	0.2960	0.2642	0.2352	0.2090	0.1854	0.1642	0.1453	0.1284	0.1134	0.1000	0.0881
-0.690	0.2869	0.2553	0.2267	0.2009	0.1778	0.1570	0.1385	0.1221	0.1075	0.0945	0.0831
-0.705	0.2779	0.2467	0.2185	0.1931	0.1704	0.1501	0.1321	0.1161	0.1019	0.0894	0.0783
-0.720	0.2692	0.2384	0.2105	0.1856	0.1633	0.1435	0.1259	0.1103	0.0966	0.0845	0.0738
-0.735	0.2608	0.2303	0.2028	0.1783	0.1565	0.1371	0.1200	0.1048	0.0915	0.0798	0.0695
-0.750	0.2525	0.2224	0.1954	0.1713	0.1499	0.1310	0.1143	0.0996	0.0867	0.0754	0.0655

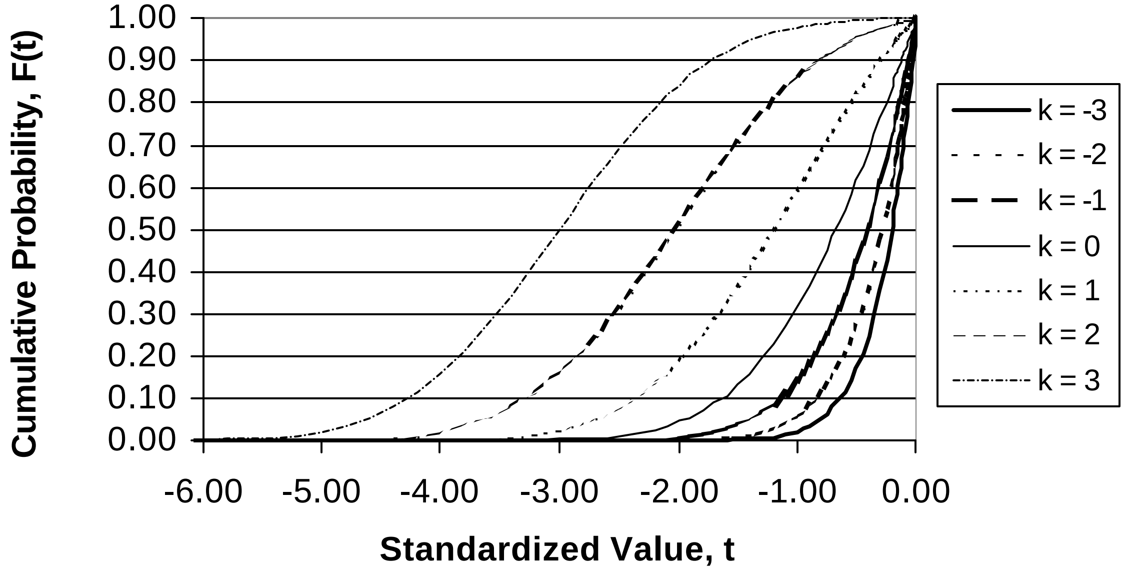

Figure 4. Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution for Standardized t's of 0 to –6

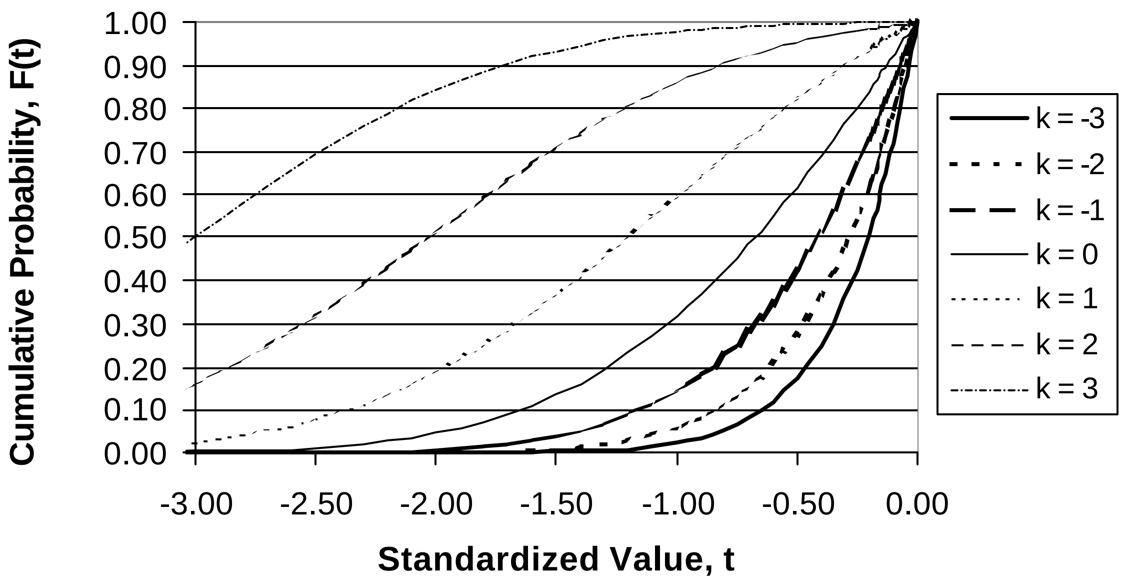

Figure 5. Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution for Standardized t's of 0 to -3

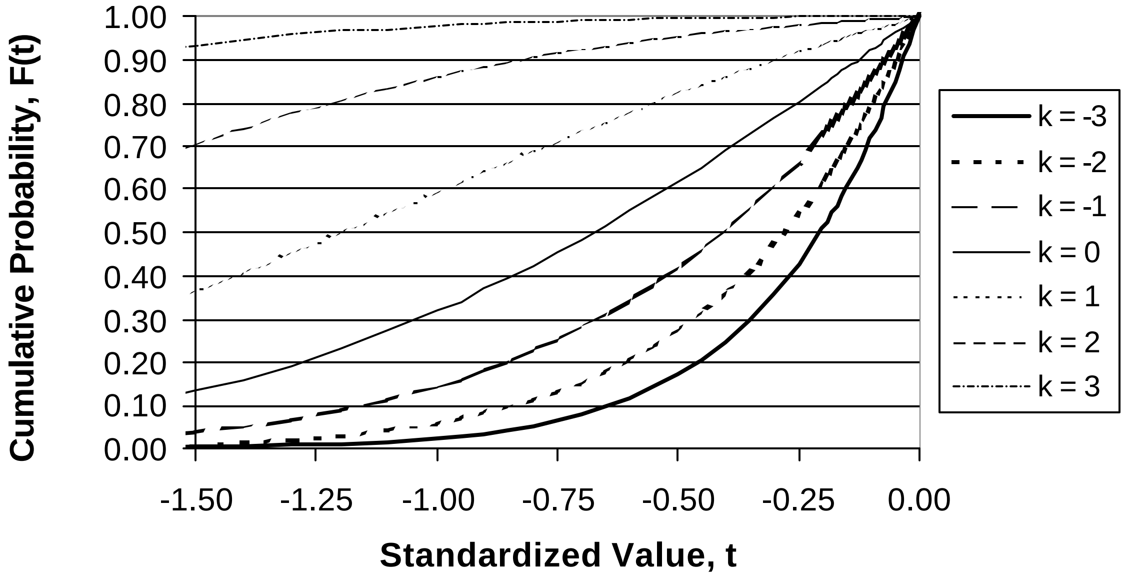

Figure 6. Cumulative Distribution Function of the Standardized, Right-Truncated Normal Distribution for Standardized t's of 0 to -1.5

Table 2. Standard t Value Associated with a Cumulative Distribution Value F(t) for the Standardized, Right-Truncated Normal Distribution

$F(t)/k_R$	-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
0.005	-1.352	-1.410	-1.472	-1.539	-1.610	-1.686	-1.768	-1.856	-1.950	-2.051	-2.158	-2.273	-2.395	-2.525	-2.662	-2.807
0.010	-1.198	-1.251	-1.308	-1.369	-1.435	-1.506	-1.582	-1.665	-1.753	-1.848	-1.950	-2.060	-2.177	-2.302	-2.435	-2.576
0.020	-1.038	-1.085	-1.137	-1.192	-1.252	-1.317	-1.387	-1.463	-1.545	-1.634	-1.729	-1.833	-1.944	-2.063	-2.190	-2.326
0.025	-0.985	-1.031	-1.080	-1.134	-1.192	-1.254	-1.322	-1.396	-1.475	-1.562	-1.655	-1.756	-1.865	-1.982	-2.107	-2.241
0.030	-0.942	-0.986	-1.034	-1.085	-1.141	-1.202	-1.268	-1.339	-1.417	-1.501	-1.593	-1.692	-1.799	-1.914	-2.038	-2.170
0.040	-0.872	-0.914	-0.959	-1.007	-1.061	-1.118	-1.181	-1.249	-1.323	-1.404	-1.492	-1.588	-1.691	-1.803	-1.924	-2.054
0.050	-0.817	-0.857	-0.900	-0.946	-0.997	-1.052	-1.112	-1.177	-1.249	-1.327	-1.412	-1.505	-1.605	-1.715	-1.833	-1.960
0.075	-0.716	-0.751	-0.790	-0.832	-0.878	-0.928	-0.983	-1.043	-1.109	-1.181	-1.260	-1.347	-1.442	-1.546	-1.658	-1.780
0.100	-0.643	-0.675	-0.710	-0.749	-0.791	-0.837	-0.888	-0.944	-1.005	-1.073	-1.148	-1.230	-1.320	-1.419	-1.527	-1.645
0.150	-0.537	-0.565	-0.595	-0.628	-0.665	-0.705	-0.750	-0.799	-0.854	-0.914	-0.981	-1.055	-1.138	-1.229	-1.329	-1.440
0.200	-0.460	-0.484	-0.511	-0.540	-0.573	-0.608	-0.648	-0.692	-0.741	-0.795	-0.856	-0.924	-1.000	-1.084	-1.178	-1.282
0.250	-0.400	-0.421	-0.445	-0.470	-0.499	-0.531	-0.566	-0.606	-0.650	-0.699	-0.755	-0.817	-0.887	-0.965	-1.053	-1.150
0.300	-0.349	-0.368	-0.389	-0.412	-0.438	-0.466	-0.498	-0.534	-0.573	-0.618	-0.669	-0.726	-0.790	-0.863	-0.944	-1.036
0.400	-0.269	-0.284	-0.300	-0.318	-0.339	-0.362	-0.387	-0.416	-0.448	-0.485	-0.526	-0.574	-0.628	-0.690	-0.761	-0.842
0.500	-0.205	-0.217	-0.230	-0.244	-0.260	-0.278	-0.298	-0.320	-0.346	-0.376	-0.410	-0.448	-0.493	-0.545	-0.605	-0.674
0.600	-0.152	-0.161	-0.171	-0.182	-0.194	-0.207	-0.223	-0.240	-0.260	-0.283	-0.309	-0.340	-0.376	-0.418	-0.467	-0.524
0.700	-0.107	-0.113	-0.120	-0.128	-0.137	-0.146	-0.157	-0.170	-0.185	-0.201	-0.221	-0.244	-0.271	-0.302	-0.340	-0.385
0.750	-0.087	-0.092	-0.097	-0.104	-0.111	-0.119	-0.128	-0.138	-0.150	-0.164	-0.180	-0.199	-0.221	-0.248	-0.280	-0.319
0.800	-0.067	-0.071	-0.076	-0.081	-0.086	-0.092	-0.100	-0.108	-0.117	-0.128	-0.141	-0.156	-0.174	-0.196	-0.222	-0.253
0.850	-0.049	-0.052	-0.055	-0.059	-0.063	-0.068	-0.073	-0.079	-0.086	-0.094	-0.104	-0.115	-0.129	-0.145	-0.165	-0.189
0.900	-0.032	-0.034	-0.036	-0.038	-0.041	-0.044	-0.048	-0.051	-0.056	-0.062	-0.068	-0.075	-0.085	-0.096	-0.109	-0.126
0.925	-0.024	-0.025	-0.027	-0.028	-0.030	-0.033	-0.035	-0.038	-0.042	-0.046	-0.050	-0.056	-0.063	-0.071	-0.081	-0.094
0.950	-0.016	-0.017	-0.018	-0.019	-0.020	-0.022	-0.023	-0.025	-0.027	-0.030	-0.033	-0.037	-0.042	-0.047	-0.054	-0.063
0.960	-0.012	-0.013	-0.014	-0.015	-0.016	-0.017	-0.019	-0.020	-0.022	-0.024	-0.027	-0.030	-0.033	-0.038	-0.043	-0.050
0.970	-0.009	-0.010	-0.010	-0.011	-0.012	-0.013	-0.014	-0.015	-0.016	-0.018	-0.020	-0.022	-0.025	-0.028	-0.032	-0.038
0.975	-0.008	-0.008	-0.009	-0.009	-0.010	-0.011	-0.011	-0.012	-0.014	-0.015	-0.017	-0.018	-0.021	-0.024	-0.027	-0.031
0.980	-0.006	-0.007	-0.007	-0.007	-0.008	-0.009	-0.009	-0.010	-0.011	-0.012	-0.013	-0.015	-0.017	-0.019	-0.022	-0.025
0.990	-0.003	-0.003	-0.003	-0.004	-0.004	-0.004	-0.005	-0.005	-0.005	-0.006	-0.007	-0.007	-0.008	-0.009	-0.011	-0.013
0.995	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	-0.003	-0.003	-0.003	-0.004	-0.004	-0.005	-0.005	-0.006
Mean	-0.283	-0.298	-0.314	-0.332	-0.352	-0.373	-0.397	-0.424	-0.454	-0.488	-0.525	-0.567	-0.615	-0.669	-0.729	-0.798
Std Dev	0.267	0.278	0.291	0.306	0.321	0.338	0.356	0.376	0.398	0.421	0.446	0.473	0.503	0.534	0.568	0.603
CoV	-0.943	-0.935	-0.928	-0.921	-0.914	-0.906	-0.897	-0.887	-0.876	-0.863	-0.850	-0.834	-0.817	-0.799	-0.778	-0.756

Table 2. (continued)

$F(t)/k_L$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.005	-2.807	-2.959	-3.119	-3.285	-3.457	-3.635	-3.818	-4.005	-4.195	-4.388	-4.584	-4.781	-4.979	-5.177	-5.377	-5.576
0.010	-2.576	-2.725	-2.881	-3.044	-3.214	-3.390	-3.572	-3.758	-3.947	-4.140	-4.335	-4.532	-4.729	-4.928	-5.127	-5.327
0.020	-2.326	-2.471	-2.623	-2.783	-2.950	-3.124	-3.304	-3.488	-3.677	-3.869	-4.063	-4.260	-4.457	-4.656	-4.855	-5.054
0.025	-2.241	-2.384	-2.535	-2.694	-2.860	-3.033	-3.212	-3.396	-3.584	-3.776	-3.970	-4.166	-4.363	-4.562	-4.761	-4.961
0.030	-2.170	-2.311	-2.461	-2.618	-2.784	-2.956	-3.134	-3.318	-3.506	-3.697	-3.891	-4.087	-4.284	-4.483	-4.682	-4.881
0.040	-2.054	-2.192	-2.340	-2.495	-2.659	-2.830	-3.007	-3.189	-3.377	-3.568	-3.761	-3.957	-4.155	-4.353	-4.552	-4.751
0.050	-1.960	-2.096	-2.242	-2.395	-2.558	-2.727	-2.903	-3.085	-3.272	-3.463	-3.656	-3.852	-4.049	-4.247	-4.446	-4.646
0.075	-1.780	-1.912	-2.053	-2.203	-2.362	-2.529	-2.703	-2.884	-3.069	-3.259	-3.452	-3.647	-3.844	-4.042	-4.241	-4.440
0.100	-1.645	-1.772	-1.910	-2.057	-2.213	-2.378	-2.550	-2.729	-2.913	-3.102	-3.295	-3.490	-3.686	-3.884	-4.083	-4.282
0.150	-1.440	-1.560	-1.691	-1.833	-1.984	-2.145	-2.314	-2.490	-2.672	-2.860	-3.051	-3.245	-3.442	-3.639	-3.838	-4.037
0.200	-1.282	-1.396	-1.521	-1.657	-1.804	-1.961	-2.127	-2.301	-2.481	-2.668	-2.858	-3.052	-3.247	-3.445	-3.643	-3.843
0.250	-1.150	-1.259	-1.379	-1.510	-1.652	-1.805	-1.968	-2.139	-2.318	-2.503	-2.692	-2.885	-3.081	-3.278	-3.476	-3.676
0.300	-1.036	-1.139	-1.254	-1.380	-1.518	-1.667	-1.827	-1.995	-2.172	-2.356	-2.544	-2.736	-2.931	-3.128	-3.327	-3.526
0.400	-0.842	-0.933	-1.037	-1.153	-1.281	-1.422	-1.575	-1.738	-1.911	-2.091	-2.277	-2.468	-2.662	-2.858	-3.056	-3.255
0.500	-0.674	-0.754	-0.846	-0.951	-1.069	-1.200	-1.345	-1.501	-1.669	-1.845	-2.029	-2.217	-2.410	-2.606	-2.803	-3.002
0.600	-0.524	-0.592	-0.671	-0.763	-0.868	-0.988	-1.122	-1.270	-1.431	-1.602	-1.782	-1.968	-2.159	-2.354	-2.551	-2.749
0.700	-0.385	-0.439	-0.503	-0.580	-0.670	-0.775	-0.896	-1.032	-1.183	-1.347	-1.521	-1.703	-1.892	-2.085	-2.281	-2.478
0.750	-0.319	-0.365	-0.421	-0.489	-0.570	-0.665	-0.777	-0.906	-1.050	-1.208	-1.378	-1.558	-1.745	-1.936	-2.132	-2.329
0.800	-0.253	-0.292	-0.339	-0.397	-0.467	-0.552	-0.653	-0.771	-0.906	-1.057	-1.222	-1.397	-1.582	-1.772	-1.966	-2.162
0.850	-0.189	-0.219	-0.256	-0.303	-0.360	-0.432	-0.519	-0.623	-0.746	-0.887	-1.043	-1.213	-1.393	-1.580	-1.773	-1.968
0.900	-0.126	-0.147	-0.173	-0.206	-0.249	-0.303	-0.371	-0.456	-0.561	-0.685	-0.827	-0.987	-1.159	-1.342	-1.531	-1.725
0.925	-0.094	-0.110	-0.130	-0.156	-0.190	-0.234	-0.290	-0.362	-0.453	-0.564	-0.696	-0.846	-1.012	-1.190	-1.377	-1.569
0.950	-0.063	-0.074	-0.088	-0.106	-0.129	-0.161	-0.203	-0.258	-0.330	-0.422	-0.536	-0.672	-0.826	-0.997	-1.178	-1.367
0.960	-0.050	-0.059	-0.070	-0.085	-0.105	-0.131	-0.166	-0.213	-0.275	-0.357	-0.460	-0.587	-0.734	-0.899	-1.077	-1.264
0.970	-0.038	-0.044	-0.053	-0.064	-0.079	-0.099	-0.127	-0.165	-0.216	-0.285	-0.375	-0.488	-0.625	-0.782	-0.954	-1.138
0.975	-0.031	-0.037	-0.044	-0.054	-0.066	-0.083	-0.107	-0.139	-0.184	-0.245	-0.327	-0.432	-0.561	-0.712	-0.881	-1.062
0.980	-0.025	-0.030	-0.035	-0.043	-0.053	-0.067	-0.087	-0.114	-0.151	-0.204	-0.275	-0.370	-0.489	-0.633	-0.795	-0.973
0.990	-0.013	-0.015	-0.018	-0.022	-0.027	-0.034	-0.044	-0.059	-0.080	-0.111	-0.155	-0.218	-0.306	-0.420	-0.560	-0.721
0.995	-0.006	-0.007	-0.009	-0.011	-0.014	-0.017	-0.022	-0.030	-0.041	-0.058	-0.083	-0.122	-0.178	-0.260	-0.370	-0.508
Mean	-0.798	-0.875	-0.962	-1.059	-1.168	-1.288	-1.419	-1.563	-1.717	-1.882	-2.055	-2.236	-2.423	-2.614	-2.808	-3.004
Std Dev	0.603	0.640	0.678	0.717	0.756	0.794	0.830	0.863	0.894	0.920	0.942	0.959	0.972	0.982	0.989	0.993
CoV	-0.756	-0.731	-0.705	-0.677	-0.647	-0.616	-0.585	-0.552	-0.520	-0.489	-0.458	-0.429	-0.401	-0.376	-0.352	-0.331

As with Table 1, use of Table 2 requires only knowledge of the truncation point k_R. Again, detailed examples to illustrate the use of Table 2 will be presented at the end of this paper.

In addition to the standardized t values associated with different values of the cumulative distribution $F_{SRTN}(t)$ – for a given point of truncation k_R – Table 2 also identifies the mean (μ_t) and standard deviation (σ_t) associated with each truncation point. These values, which were calculated using Equations 13 and 17, respectively, were then used to calculate the coefficient of variation c associated with each truncation point. As noted earlier, this coefficient of variation is uniquely determined for the standardized, right-truncated normal distribution by the value of the truncation point k_R. It is worth noting that, due to the definition of the standardized t value for the right-truncated normal distribution, the mean (μ_t) is negative – as is the coefficient of variation c.

For situations in which the truncation point is not precisely known, it can be estimated using order statistics, and then Tables 1 and 2 can be utilized. Alternatively, the sample mean and standard deviation can be used to estimate the coefficient of variation as follows

$$c = \frac{\sigma_t}{\mu_t} \approx \frac{s}{\bar{x}} \tag{21}$$

where the accuracy of this approximation will improve as the sample size increases. This estimated coefficient of variation c can then be used along with Table 2 to identify the approximate standardized truncation point k_R – subject to the granularity of the coefficients of variation presented in Table 2.

To assist in such estimation of the truncation point, Table 3 summarizes the truncation point k_R associated with coefficients of variation c over the range -0.2(-0.02)-1.68 – which were determined using a dichotomous line search algorithm. In addition, Figure 7 plots the coefficient of variation c as a function of the point of truncation k_R.

In utilizing Table 2 or 3 based upon available sample statistics, the user must recognize that, due to right truncation, the coefficient of variation c and mean μ_t of the standardized, right-truncated normal distribution are negative.

Table 3. Associated Values of the Coefficient of Variation c and Truncation Point k_R for the Standardized, Right-Truncated Normal Distribution

c	k_R	c	k_R	c	k_R	c	k_R	c	k_R
-0.20	5.00	-0.50	1.73	-0.80	-0.41	-1.10	-4.18	-1.40	-4.80
-0.22	4.55	-0.52	1.60	-0.82	-0.63	-1.12	-4.24	-1.42	-4.83
-0.24	4.17	-0.54	1.48	-0.84	-0.87	-1.14	-4.30	-1.44	-4.86
-0.26	3.84	-0.56	1.35	-0.86	-1.15	-1.16	-4.35	-1.46	-4.88
-0.28	3.57	-0.58	1.23	-0.88	-1.47	-1.18	-4.40	-1.48	-4.91
-0.30	3.32	-0.60	1.10	-0.90	-1.87	-1.20	-4.45	-1.50	-4.94
-0.32	3.11	-0.62	0.98	-0.92	-2.38	-1.22	-4.49	-1.52	-4.96
-0.34	2.91	-0.64	0.85	-0.94	-2.93	-1.24	-4.53	-1.54	-4.99
-0.36	2.73	-0.66	0.71	-0.96	-3.31	-1.26	-4.57	-1.56	-5.01
-0.38	2.57	-0.68	0.58	-0.98	-3.55	-1.28	-4.61	-1.58	-5.03
-0.40	2.41	-0.70	0.43	-1.00	-3.71	-1.30	-4.64	-1.60	-5.05
-0.42	2.26	-0.72	0.29	-1.02	-3.84	-1.32	-4.68	-1.62	-5.08
-0.44	2.12	-0.74	0.13	-1.04	-3.94	-1.34	-4.71	-1.64	-5.10
-0.46	1.99	-0.76	-0.04	-1.06	-4.03	-1.36	-4.74	-1.66	-5.12
-0.48	1.86	-0.78	-0.22	-1.08	-4.11	-1.38	-4.77	-1.68	-5.14

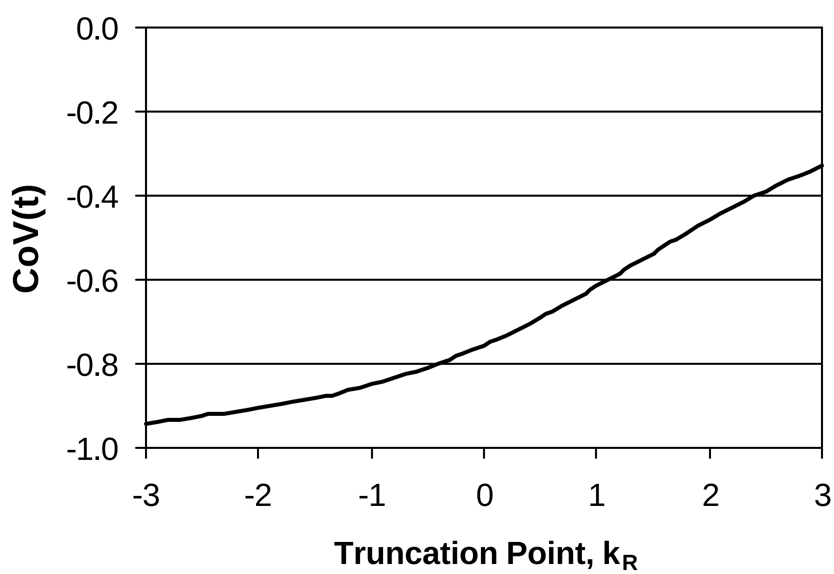

Figure 7. Coefficient of Variation as a Function of Truncation Point k_R for the Standardized, Right-Truncated Normal Distribution

Examples of Use of the Tables for the Standardized, Right-Truncated Normal Distribution

Table 1. A service facility manager is developing a plan for the staffing of a facility whose hourly number of customers desiring service is normally distributed with a mean of 20 customers and a standard deviation of 5 customers. Specifically, he is interested in determining the probability that the number of customers desiring service in a particular hour exceeds 10 customers. Due to waiting area limitations, the number of customers desiring service at any time cannot exceed 27 customers. If more than 27 customers desire service, those who cannot find a space in the waiting area go elsewhere and are effectively lost to the service facility – i.e., the actual number of customers desiring service is truncated at values above 27 customers.

Clearly, since more than 27 customers cannot be serviced, this demand pattern would be most appropriately modeled as that of a right-truncated normal distribution. Using the notation of Figure 1: $\mu = 20$, $\sigma = 5$, and $x_R = 27$. In terms of the standard normal distribution, the truncation point k_R can be calculated using Equation 5

$$k_R = \frac{x_R - \mu}{\sigma} = \frac{27 - 20}{5} = 1.4$$

Finally, to develop the standardized, right-truncated normal distribution for this demand pattern, use the definitional relation $t = z - k_R$. In this case, $t = z - 1.4$.

The service facility manager's interest in greater than 10 customers within a given one-hour period ($P(x \geq 10)$), then, has a standard normal equivalent of $P(z \geq -2)$ and a standardized, right-truncated normal equivalent of $P(t \geq -3.4)$.

$$P(x \geq 10) = P\left(z \geq \frac{10 - 20}{5}\right) = P(z \geq -2)$$
$$= P(t \geq -2 - 1.4) = P(t \geq -3.4)$$

Using Table 1, the cumulative distribution function associated with a standardized t value of –3.4 and truncation point k_R **=** 1.4 can be found to be $F(t = -3.4) = 0.0247$. Thus, the probability that the number of customers within a one-hour period will exceed 10 is given by

$$P(x \geq 10) = P(t \geq -3.4) = 1 - F(t = -3.4)_{k_R = 1.4} = 1 - 0.0247 = 0.9753$$

In other words, there is a 97.53% chance that more than 10 customers will desire service during a one-hour period. Contrast this with the value of 97.72% that would be obtained using tables of the standard (non-truncated) normal distribution.

<u>Table 2</u>. Continuing the service facility management example above, suppose that the manager desires to insure that enough staff is on hand to satisfy 90% of the likely number of customers during a one-hour period. What number of customers corresponds to the 90th percentile of the expected demand?

Using Table 2, for a truncation point of $k_R = 1.4$, it can be seen that achieving $F_{SRTN}(t) = 0.90$ requires a standardized t value of t = -0.4564. This, then, implies a standard normal value z of

$$z = t + k = -0.4564 + 1.4 = 0.9436$$

and a "real-world" number of customers of

$$x = \mu + z\sigma = 20 + 0.9436(5) = 24.72$$

In other words, the manager should plan for up to 24.72 customers in order to be able to meet the 90th percentile. Again, this value can be contrasted with the one of 26.40 customers that would be obtained using non-truncated distributions.

The difference in these values, although small for this example, illustrates the fact that the Table 4-2 values account for the fact that the mean of the truncated distribution is actually 19.19 customers (slightly less than the non-truncated distribution) and its standard deviation is actually 4.32 customers (also slightly less than the non-truncated).

Table 3. The facility manager now needs to develop a staffing plan for another one-hour period at a different facility (presumably, in which the maximum number of customers possible is unknown) based upon a large ($n \geq 30$) sample of hourly periods – which have a sample mean of 20 customers and a sample standard deviation of 8 customers. Based upon these sample data, what would be the appropriate point of truncation to assume?

Using Equation 21, the coefficient of variation can be estimated to be $c = 0.40$. From Table 3, this implies a truncation point of $k_R = 2.41$ – where one looks up a (negative) value of $c = -0.40$ since the truncation point is assumed to be to the right.

Reference Visualizations for the Standardized, Right-Truncated Normal Distribution

This section presents visualizations of the standardized, right-truncated normal distribution for reference in working with this family of distributions. Specifically – for upper truncations points k_R of -3, -2, -1, 0, 1, 2, and 3 – this paper provides figures that graphically represent the probability density function $f_{SRTN}(t)$ and the cumulative probability distribution $F_{SRTN}(t)$ for the appropriate range of standard t values. In addition, the mean, standard deviation, and coefficient of variation are summarized for each of these figures – based upon the values determined in Table 2.

In Figures 8 through 14, the dashed line is that of the probability density function $f_{SRTN}(t)$ and the solid line is that of the cumulative distribution function $F_{SRTN}(t)$.

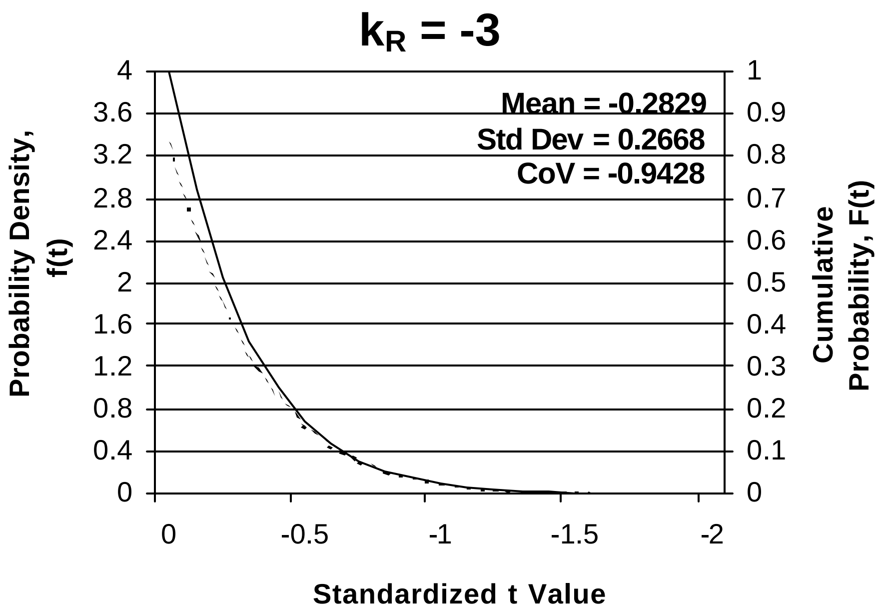

Figure 8. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = -3

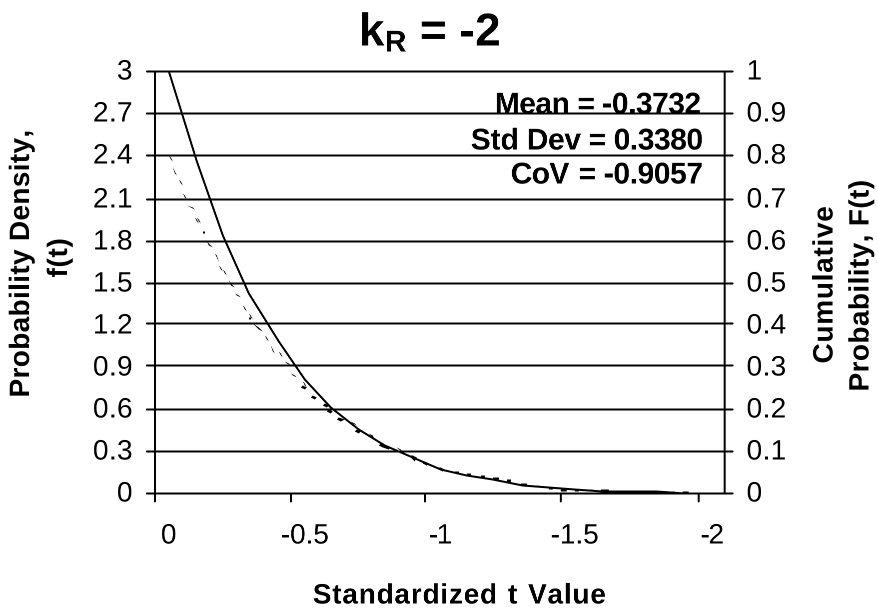

Figure 9. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = -2

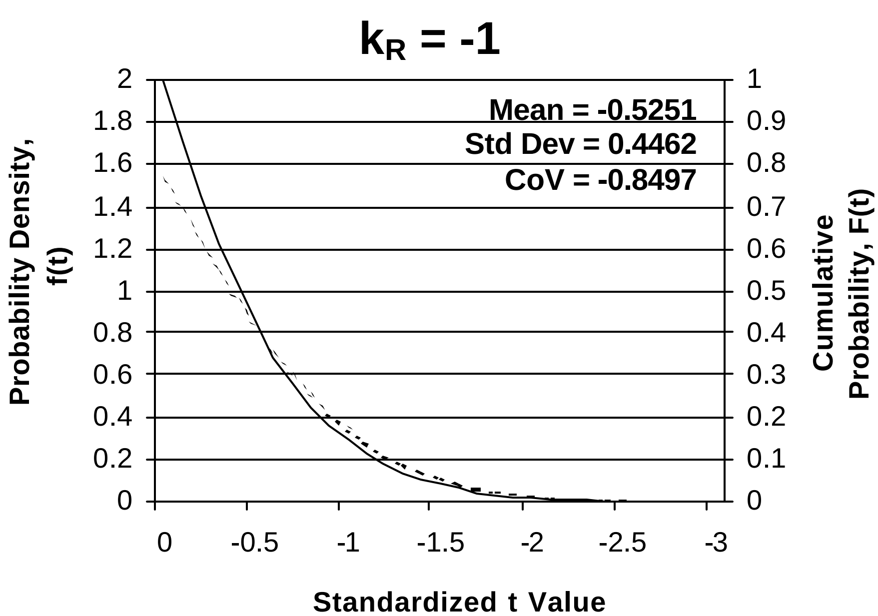

Figure 10. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = -1

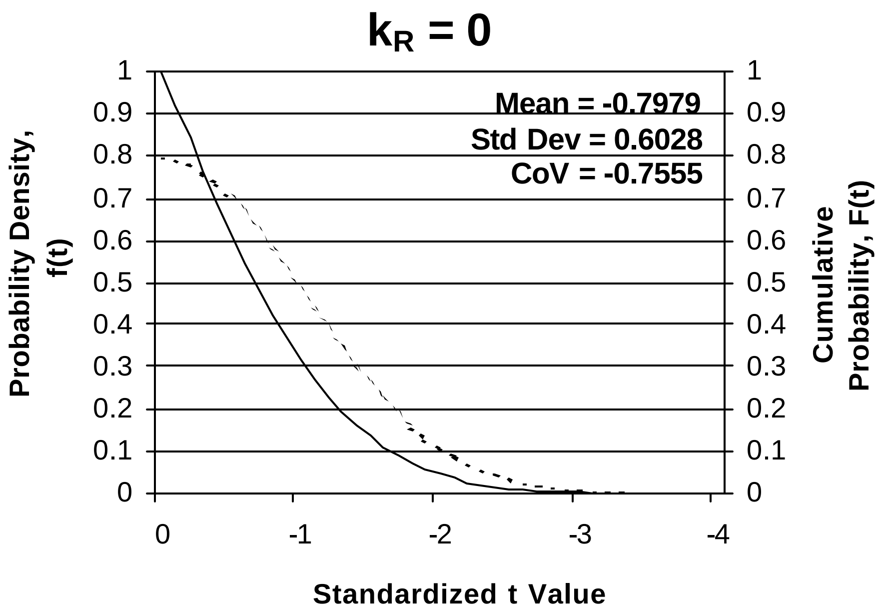

Figure 11. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = 0

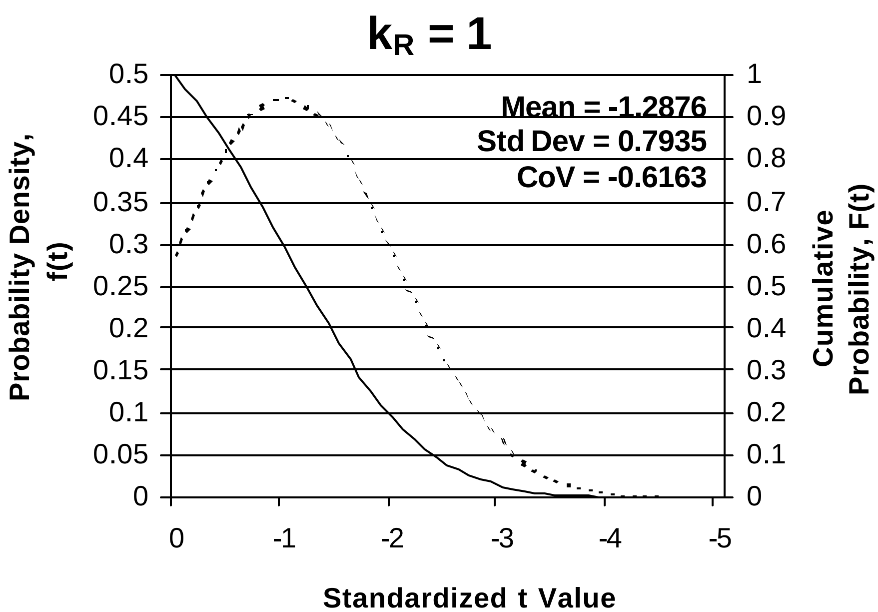

Figure 12. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = 1

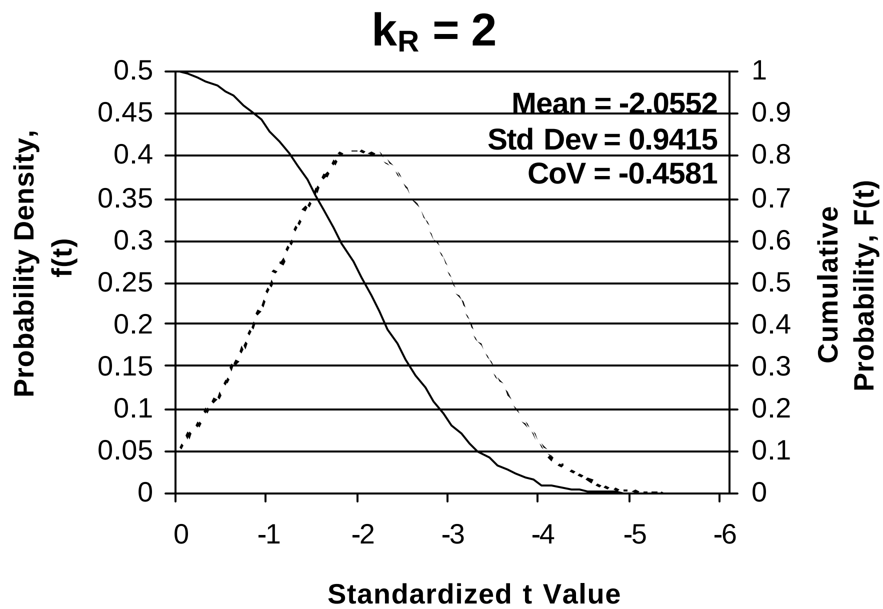

Figure 13. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = 2

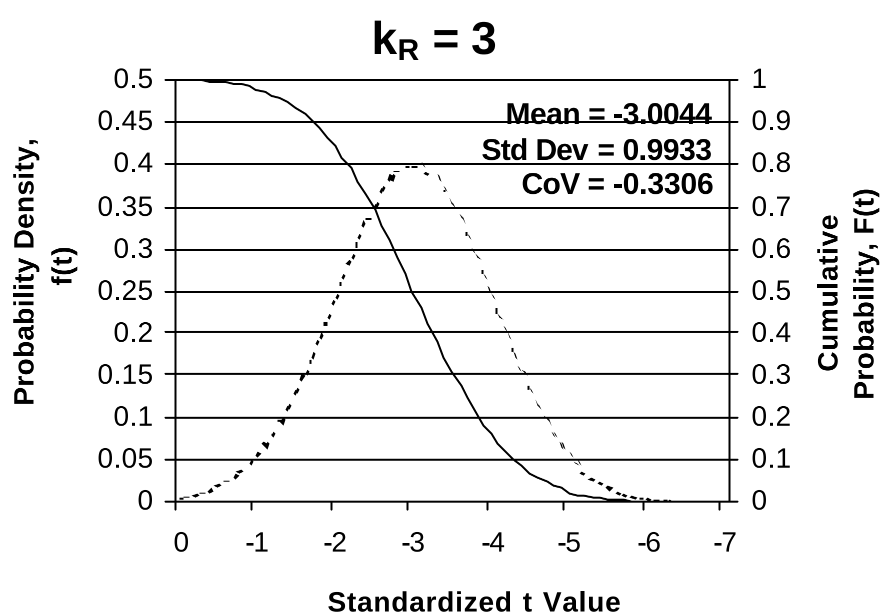

Figure 14. Visualization of the Standardized, Right-Truncated Normal Distribution with Point of Truncation k_R = 3

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