

**REFORMULATION IN TERMS OF THE STANDARD NORMAL VARIATE**

Arvid C. Johnson, Dominican University, River Forest, IL

This paper provides a reformulation of selected key tables for the left-, right-, and doubly-truncated normal distribution in terms of the standard normal variate ( $z$ ). Many workers/researchers may find these reformulations more attractive than those presented elsewhere due to their similarity to tables used for the non-truncated normal distribution. Unfortunately, the coefficient of variation is, by definition, undefined for cases of symmetric (and light) truncation.

Recall that, in terms of the standard normal variable ( $z$ ), where

$$z = \frac{x - \mu}{\sigma} \quad (1)$$

the equation for the non-truncated normal distribution is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty \leq z \leq \infty \quad (2)$$

### Left-Truncated Normal Distribution

**Definition.** For the case of a left-truncated normal distribution - i.e., one in which values of the normal variable below some point  $x_L$  cannot be observed - the point of truncation  $x_L$  can be defined in terms of the standard normal variable and will be denoted  $k_L$  as given by

$$k_L = \frac{x_L - \mu}{\sigma} \quad (3)$$

Thus, the left-truncated normal distribution can be represented in terms of the standard normal distribution by the following: (See Figure 1.)

$$f_{L\text{TN}}(z) = \begin{cases} 0, & -\infty \leq z \leq k_L \\ \frac{f(z)}{\int_{k_L}^{\infty} f(z)dz}, & k_L \leq z \leq \infty \end{cases} \quad (4)$$

From Equation 4, one may then find the cumulative distribution function of the left-truncated normal distribution, as shown below:

$$F_{L\text{TN}}(z) = \begin{cases} 0, & z \leq k_L \\ \frac{F(z) - F(k_L)}{1 - F(k_L)}, & z \geq k_L \end{cases} \quad (5)$$

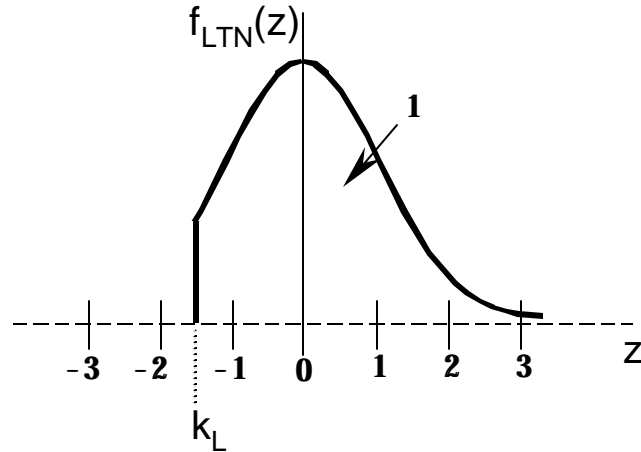


Figure 1. Left-Truncated Normal Distribution (in Terms of the Standard Normal Distribution)

where  $f(z)$  is as defined in Equation 2, and  $F(z)$  is given by

$$F(k) = \int_{-\infty}^k f(z) dz \quad (6)$$

**Characteristic Parameters.** The mean ( $\mu_z$ ) and standard deviation ( $\sigma_z$ ) of the left-truncated normal distribution - as defined in Equation 4 - for a given point of truncation  $k_L$  can be found in [Joh70] and elsewhere.

$$\mu_z = \frac{f(k_L)}{1 - F(k_L)} \quad (7)$$

$$\sigma_z = \sqrt{E(z^2) - \mu_z^2} \quad (8)$$

where

$$E(z^2) = \frac{1}{1 - F(k_L)} [(1 - F(k_L)) + k_L f(k_L)]$$

As noted elsewhere, the point of truncation  $k_L$  solely and uniquely defines the mean and standard deviation of the left-truncated normal distribution. As before, the coefficient of variation  $c$  - which exists uniquely for a particular  $k_L$  - can be defined as

$$c = \frac{\sigma_z}{\mu_z} \quad (9)$$

**Reformulated Tables.** Using Equations 5 through 9, a prior table has been reformulated in terms of the standard normal variate ( $z$ ). This reformulated table is presented as Table 1 and provides the standard normal variate ( $z$ ) associated with a given value of the cumulative distribution function  $F(z)$  over the range  $F(z) = 0.005$  to  $F(z) = 0.995$ , with varying granularity. Table 1 is presented in two segments - for negative and positive values, respectively, of the point of truncation  $k_L$  over the range  $-3.0(0.2)3.0$ .

As may be noted in a review of Table 1, the coefficient of variation - as defined in Equation 9 - approaches infinity as the degree of truncation is reduced and approaches zero for heavy truncation. This is illustrated in Figure 2, in which the coefficients of variation in terms of the standardized truncated value  $t$

Table 1. Standard Normal Variate  $z$  Associated with a Cumulative Distribution Value  $F(z)$  for the Left-Truncated Normal Distribution

$F(z)/K_0$	-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
0.005	-2.492	-2.430	-2.340	-2.222	-2.078	-1.917	-1.742	-1.559	-1.370	-1.178	-0.983	-0.786	-0.589	-0.391	-0.193	0.006
0.010	-2.279	-2.240	-2.180	-2.094	-1.982	-1.845	-1.689	-1.520	-1.341	-1.156	-0.966	-0.773	-0.578	-0.382	-0.185	0.013
0.020	-2.027	-2.005	-1.967	-1.911	-1.830	-1.725	-1.596	-1.449	-1.286	-1.113	-0.933	-0.747	-0.557	-0.365	-0.170	0.025
0.025	-1.938	-1.919	-1.888	-1.839	-1.768	-1.673	-1.555	-1.416	-1.261	-1.093	-0.917	-0.734	-0.546	-0.356	-0.163	0.031
0.030	-1.862	-1.846	-1.818	-1.775	-1.712	-1.625	-1.515	-1.384	-1.235	-1.073	-0.901	-0.721	-0.536	-0.347	-0.156	0.038
0.040	-1.736	-1.723	-1.701	-1.666	-1.613	-1.540	-1.443	-1.325	-1.187	-1.034	-0.869	-0.695	-0.515	-0.330	-0.141	0.050
0.050	-1.633	-1.622	-1.603	-1.574	-1.528	-1.464	-1.378	-1.270	-1.142	-0.997	-0.839	-0.671	-0.494	-0.312	-0.126	0.063
0.075	-1.431	-1.423	-1.410	-1.388	-1.354	-1.304	-1.236	-1.147	-1.038	-0.910	-0.766	-0.610	-0.444	-0.270	-0.090	0.094
0.100	-1.275	-1.269	-1.258	-1.241	-1.213	-1.173	-1.115	-1.039	-0.944	-0.829	-0.697	-0.551	-0.394	-0.227	-0.053	0.126
0.150	-1.032	-1.027	-1.020	-1.007	-0.987	-0.957	-0.913	-0.854	-0.777	-0.681	-0.568	-0.440	-0.297	-0.144	0.019	0.189
0.200	-0.838	-0.834	-0.828	-0.818	-0.803	-0.778	-0.743	-0.694	-0.629	-0.547	-0.448	-0.333	-0.203	-0.061	0.092	0.253
0.250	-0.671	-0.668	-0.664	-0.655	-0.642	-0.622	-0.592	-0.550	-0.494	-0.423	-0.335	-0.230	-0.111	0.021	0.165	0.319
0.300	-0.522	-0.519	-0.515	-0.508	-0.497	-0.479	-0.453	-0.417	-0.368	-0.304	-0.225	-0.130	-0.020	0.103	0.239	0.385
0.400	-0.251	-0.249	-0.246	-0.241	-0.232	-0.218	-0.198	-0.169	-0.130	-0.078	-0.012	0.068	0.163	0.271	0.392	0.524
0.500	0.002	0.003	0.006	0.010	0.017	0.029	0.045	0.069	0.101	0.145	0.200	0.269	0.351	0.446	0.554	0.674
0.600	0.255	0.256	0.258	0.262	0.268	0.277	0.291	0.311	0.338	0.375	0.422	0.481	0.553	0.637	0.733	0.842
0.700	0.526	0.527	0.528	0.531	0.536	0.544	0.556	0.572	0.595	0.627	0.667	0.718	0.780	0.854	0.939	1.036
0.750	0.676	0.676	0.678	0.681	0.685	0.692	0.703	0.718	0.739	0.768	0.805	0.852	0.910	0.979	1.059	1.150
0.800	0.843	0.843	0.845	0.847	0.852	0.858	0.868	0.881	0.901	0.927	0.961	1.004	1.057	1.121	1.196	1.282
0.850	1.037	1.038	1.039	1.042	1.045	1.051	1.060	1.072	1.090	1.114	1.145	1.184	1.233	1.291	1.360	1.440
0.900	1.282	1.283	1.284	1.286	1.290	1.295	1.302	1.313	1.329	1.350	1.378	1.413	1.457	1.510	1.572	1.645
0.925	1.440	1.441	1.442	1.444	1.447	1.452	1.459	1.469	1.484	1.503	1.529	1.562	1.603	1.653	1.712	1.780
0.950	1.646	1.646	1.647	1.649	1.652	1.656	1.663	1.672	1.685	1.703	1.727	1.758	1.795	1.842	1.896	1.960
0.960	1.751	1.752	1.753	1.755	1.757	1.761	1.768	1.777	1.789	1.807	1.830	1.859	1.895	1.940	1.992	2.054
0.970	1.881	1.882	1.883	1.884	1.887	1.891	1.897	1.906	1.918	1.934	1.956	1.984	2.018	2.061	2.111	2.170
0.975	1.961	1.961	1.962	1.963	1.966	1.970	1.976	1.984	1.996	2.012	2.033	2.060	2.094	2.135	2.184	2.241
0.980	2.054	2.055	2.056	2.057	2.060	2.063	2.069	2.077	2.088	2.104	2.124	2.150	2.183	2.223	2.271	2.326
0.990	2.327	2.327	2.328	2.329	2.332	2.335	2.340	2.347	2.358	2.372	2.390	2.414	2.444	2.481	2.525	2.576
0.995	2.576	2.577	2.577	2.579	2.581	2.584	2.588	2.595	2.605	2.618	2.635	2.657	2.685	2.719	2.759	2.807
Mean	0.004	0.008	0.014	0.023	0.036	0.055	0.082	0.117	0.163	0.219	0.288	0.368	0.459	0.562	0.675	0.798
Std Dev	0.993	0.989	0.982	0.972	0.959	0.942	0.920	0.894	0.863	0.830	0.794	0.756	0.717	0.678	0.640	0.603
CoV	223.8	124.6	71.96	43.06	26.66	17.04	11.23	7.614	5.301	3.781	2.759	2.056	1.561	1.206	0.948	0.756

Table 1. (continued)

$F(z)/K_z$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.005	0.006	0.205	0.405	0.604	0.804	1.003	1.203	1.403	1.602	1.802	2.002	2.202	2.402	2.602	2.802	3.002
0.010	0.013	0.211	0.409	0.608	0.807	1.007	1.206	1.405	1.605	1.805	2.004	2.204	2.404	2.603	2.803	3.003
0.020	0.025	0.222	0.419	0.617	0.815	1.013	1.212	1.411	1.610	1.809	2.009	2.208	2.407	2.607	2.807	3.006
0.025	0.031	0.227	0.424	0.621	0.818	1.017	1.215	1.414	1.612	1.811	2.011	2.210	2.409	2.609	2.808	3.008
0.030	0.038	0.232	0.428	0.625	0.822	1.020	1.218	1.416	1.615	1.814	2.013	2.212	2.411	2.610	2.810	3.009
0.040	0.050	0.243	0.438	0.633	0.830	1.027	1.224	1.422	1.620	1.819	2.017	2.216	2.415	2.614	2.813	3.012
0.050	0.063	0.254	0.447	0.642	0.837	1.033	1.230	1.427	1.625	1.823	2.022	2.220	2.419	2.618	2.817	3.016
0.075	0.094	0.281	0.471	0.663	0.856	1.050	1.246	1.442	1.638	1.835	2.033	2.230	2.428	2.627	2.825	3.024
0.100	0.126	0.309	0.496	0.685	0.875	1.068	1.262	1.456	1.651	1.848	2.044	2.241	2.438	2.636	2.834	3.032
0.150	0.189	0.365	0.545	0.729	0.915	1.104	1.294	1.486	1.679	1.873	2.068	2.263	2.459	2.655	2.852	3.049
0.200	0.253	0.422	0.596	0.774	0.956	1.141	1.328	1.517	1.708	1.900	2.092	2.286	2.481	2.676	2.871	3.067
0.250	0.319	0.480	0.648	0.821	0.999	1.180	1.364	1.550	1.738	1.928	2.119	2.311	2.504	2.697	2.892	3.087
0.300	0.385	0.540	0.702	0.871	1.044	1.221	1.401	1.585	1.770	1.957	2.146	2.337	2.528	2.720	2.913	3.107
0.400	0.524	0.667	0.818	0.976	1.140	1.309	1.483	1.660	1.840	2.023	2.207	2.394	2.582	2.771	2.961	3.152
0.500	0.674	0.805	0.945	1.093	1.248	1.410	1.576	1.746	1.920	2.098	2.278	2.460	2.644	2.830	3.017	3.205
0.600	0.842	0.961	1.090	1.228	1.374	1.526	1.685	1.848	2.016	2.187	2.362	2.539	2.718	2.900	3.084	3.269
0.700	1.036	1.144	1.263	1.390	1.526	1.669	1.818	1.973	2.134	2.298	2.466	2.638	2.812	2.989	3.168	3.349
0.750	1.150	1.253	1.365	1.487	1.617	1.755	1.899	2.050	2.206	2.366	2.531	2.699	2.870	3.045	3.221	3.400
0.800	1.282	1.378	1.484	1.600	1.724	1.856	1.995	2.141	2.292	2.448	2.608	2.773	2.940	3.111	3.284	3.460
0.850	1.440	1.529	1.629	1.738	1.855	1.981	2.114	2.254	2.399	2.550	2.705	2.865	3.028	3.195	3.365	3.537
0.900	1.645	1.727	1.819	1.920	2.030	2.148	2.273	2.405	2.544	2.688	2.837	2.991	3.149	3.310	3.475	3.643
0.925	1.780	1.858	1.946	2.042	2.147	2.260	2.381	2.509	2.643	2.783	2.928	3.078	3.232	3.390	3.551	3.716
0.950	1.960	2.033	2.115	2.205	2.305	2.412	2.527	2.649	2.777	2.912	3.052	3.197	3.346	3.500	3.657	3.817
0.960	2.054	2.124	2.203	2.291	2.388	2.492	2.604	2.723	2.849	2.981	3.118	3.261	3.407	3.559	3.714	3.872
0.970	2.170	2.238	2.314	2.399	2.492	2.593	2.701	2.817	2.939	3.068	3.202	3.341	3.485	3.634	3.786	3.942
0.975	2.241	2.307	2.382	2.465	2.556	2.655	2.762	2.875	2.996	3.122	3.254	3.392	3.534	3.680	3.831	3.985
0.980	2.326	2.390	2.463	2.544	2.633	2.729	2.834	2.945	3.063	3.187	3.317	3.452	3.592	3.737	3.885	4.038
0.990	2.576	2.635	2.702	2.777	2.860	2.950	3.048	3.153	3.265	3.382	3.506	3.635	3.769	3.908	4.051	4.198
0.995	2.807	2.862	2.925	2.995	3.073	3.158	3.251	3.350	3.456	3.568	3.686	3.810	3.939	4.072	4.210	4.352
Mean	0.798	0.929	1.069	1.215	1.367	1.525	1.688	1.854	2.024	2.197	2.373	2.552	2.732	2.914	3.098	3.283
Std Dev	0.603	0.568	0.534	0.503	0.473	0.446	0.421	0.398	0.376	0.356	0.338	0.321	0.306	0.291	0.278	0.267
CoV	0.756	0.611	0.500	0.414	0.346	0.293	0.249	0.214	0.186	0.162	0.142	0.126	0.112	0.100	0.090	0.081

and the standard normal variate  $z$  are plotted over the range of truncation points  $k_L$  from -3 to 3.

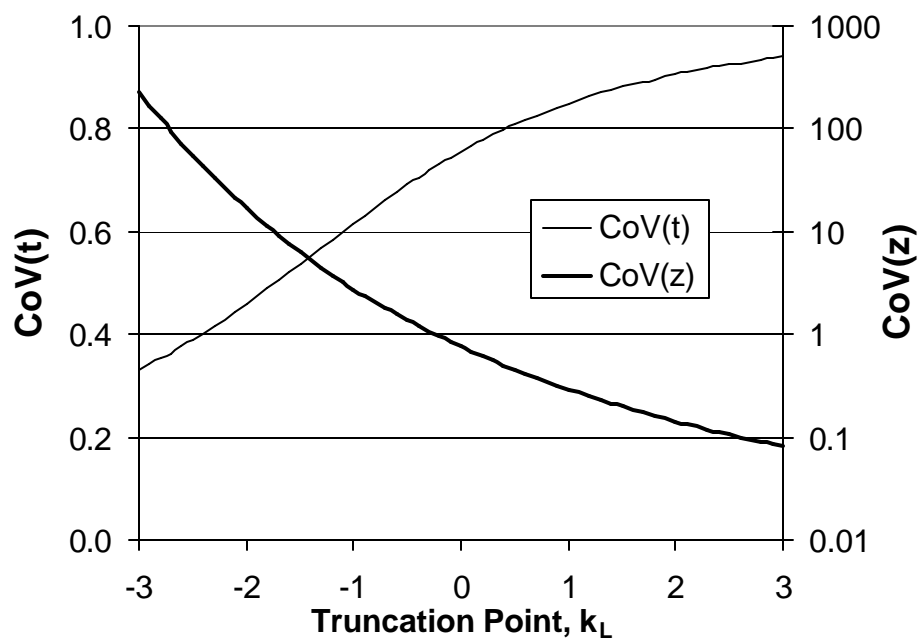


Figure 2. Coefficients of Variation in Terms of the Standardized Variate  $t$  and the Standard Normal Variate  $z$  as a Function of the Point of Truncation  $k_L$

### Right-Truncated Normal Distribution

**Definition.** For the case of a right-truncated normal distribution - i.e., one in which values of the normal variable above some point  $x_R$  cannot be observed - the point of truncation  $x_R$  can be defined in terms of the standard normal variable and will be denoted  $k_R$  as given by

$$k_R = \frac{x_R - \mu}{\sigma} \quad (10)$$

Thus, the right-truncated normal distribution can be represented in terms of the standard normal distribution by the following: (See Figure 3.)

$$f_{\text{RTN}}(z) = \begin{cases} \frac{f(z)}{\int_{-\infty}^{k_R} f(z) dz}, & -\infty \leq z \leq k_R \\ 0, & k_R \leq z \leq \infty \end{cases} \quad (11)$$

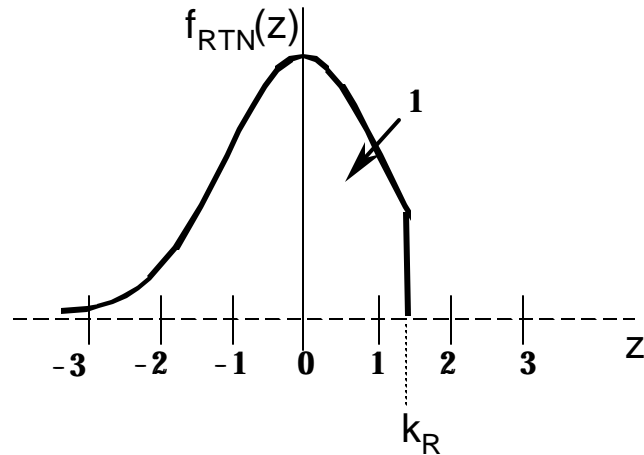


Figure 3. Right-Truncated Normal Distribution (in Terms of the Standard Normal Distribution)

From Equation 11, one may then find the cumulative distribution function of the right-truncated normal distribution, as shown below:

$$F_{\text{RTN}}(z) = \begin{cases} \frac{F(z)}{F(k_R)}, & z \leq k_R \\ 1, & z \geq k_R \end{cases} \quad (12)$$

where  $f(z)$  is as defined in Equation 2, and  $F(z)$  is given by Equation 6.

**Characteristic Parameters.** The mean ( $\mu_z$ ) and standard deviation ( $\sigma_z$ ) of the right-truncated normal distribution - as defined in Equation 11 - for a given point of truncation  $k_R$  can be found in [Joh70] and elsewhere.

$$\mu_t = -\frac{f(k_R)}{F(k_R)} \quad (13)$$

$$\sigma_z = \sqrt{E(z^2) - \mu_z^2} \quad (14)$$

where

$$E(z^2) = \frac{1}{F(k_R)} [F(k_R) - k_R f(k_R)]$$

As noted elsewhere, the point of truncation  $k_R$  solely and uniquely defines the mean and standard deviation of the left-truncated normal distribution. As before, the coefficient of variation  $c$  - which exists uniquely for a particular  $k_R$  - can be defined as

$$c = \frac{\sigma_z}{\mu_z} \quad (15)$$

**Reformulated Tables.** Using Equations 12 through 15, a prior table has been reformulated in terms of the standard normal variate ( $z$ ). This reformulated table is presented as Table 2 and provides the standard normal variate ( $z$ ) associated with a given value of the cumulative distribution function  $F(z)$  over the range  $F(z) = 0.005$  to  $F(z) = 0.995$ , with varying granularity. Table 2 is presented in two segments - for negative and positive values, respectively, of the point of truncation  $k_R$  over the range  $-3.0(0.2)3.0$ .

As may be noted in a review of Table 2, the coefficient of variation - as defined in Equation 15 - approaches infinity as the degree of truncation is reduced and approaches zero for heavy truncation. This is illustrated in Figure 4, in which the coefficients of variation in terms of the standardized truncated value  $t$  and the standard normal variate  $z$  are plotted over the range of truncation points  $k_R$  from -3 to 3.

Table 2. Standard Normal Variate  $z$  Associated with a Cumulative Distribution Value  $F(z)$  for the Right-Truncated Normal Distribution ( $k_R = -3$  to  $0$ )

$F(z)/k_R$	-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
0.005	-4.352	-4.210	-4.072	-3.939	-3.810	-3.686	-3.568	-3.456	-3.350	-3.251	-3.158	-3.073	-2.995	-2.925	-2.862	-2.807
0.010	-4.198	-4.051	-3.908	-3.769	-3.635	-3.506	-3.382	-3.265	-3.153	-3.048	-2.950	-2.860	-2.777	-2.702	-2.635	-2.576
0.020	-4.038	-3.885	-3.737	-3.592	-3.452	-3.317	-3.187	-3.063	-2.945	-2.834	-2.729	-2.633	-2.544	-2.463	-2.390	-2.326
0.025	-3.985	-3.831	-3.680	-3.534	-3.392	-3.254	-3.122	-2.996	-2.875	-2.762	-2.655	-2.556	-2.465	-2.382	-2.307	-2.241
0.030	-3.942	-3.786	-3.634	-3.485	-3.341	-3.202	-3.068	-2.939	-2.817	-2.701	-2.593	-2.492	-2.399	-2.314	-2.238	-2.170
0.040	-3.872	-3.714	-3.559	-3.407	-3.261	-3.118	-2.981	-2.849	-2.723	-2.604	-2.492	-2.388	-2.291	-2.203	-2.124	-2.054
0.050	-3.817	-3.657	-3.500	-3.346	-3.197	-3.052	-2.912	-2.777	-2.649	-2.527	-2.412	-2.305	-2.205	-2.115	-2.033	-1.960
0.075	-3.716	-3.551	-3.390	-3.232	-3.078	-2.928	-2.783	-2.643	-2.509	-2.381	-2.260	-2.147	-2.042	-1.946	-1.858	-1.780
0.100	-3.643	-3.475	-3.310	-3.149	-2.991	-2.837	-2.688	-2.544	-2.405	-2.273	-2.148	-2.030	-1.920	-1.819	-1.727	-1.645
0.150	-3.537	-3.365	-3.195	-3.028	-2.865	-2.705	-2.550	-2.399	-2.254	-2.114	-1.981	-1.855	-1.738	-1.629	-1.529	-1.440
0.200	-3.460	-3.284	-3.111	-2.940	-2.773	-2.608	-2.448	-2.292	-2.141	-1.995	-1.856	-1.724	-1.600	-1.484	-1.378	-1.282
0.250	-3.400	-3.221	-3.045	-2.870	-2.699	-2.531	-2.366	-2.206	-2.050	-1.899	-1.755	-1.617	-1.487	-1.365	-1.253	-1.150
0.300	-3.349	-3.168	-2.989	-2.812	-2.638	-2.466	-2.298	-2.134	-1.973	-1.818	-1.669	-1.526	-1.390	-1.263	-1.144	-1.036
0.400	-3.269	-3.084	-2.900	-2.718	-2.539	-2.362	-2.187	-2.016	-1.848	-1.685	-1.526	-1.374	-1.228	-1.090	-0.961	-0.842
0.500	-3.205	-3.017	-2.830	-2.644	-2.460	-2.278	-2.098	-1.920	-1.746	-1.576	-1.410	-1.248	-1.093	-0.945	-0.805	-0.674
0.600	-3.152	-2.961	-2.771	-2.582	-2.394	-2.207	-2.023	-1.840	-1.660	-1.483	-1.309	-1.140	-0.976	-0.818	-0.667	-0.524
0.700	-3.107	-2.913	-2.720	-2.528	-2.337	-2.146	-1.957	-1.770	-1.585	-1.401	-1.221	-1.044	-0.871	-0.702	-0.540	-0.385
0.750	-3.087	-2.892	-2.697	-2.504	-2.311	-2.119	-1.928	-1.738	-1.550	-1.364	-1.180	-0.999	-0.821	-0.648	-0.480	-0.319
0.800	-3.067	-2.871	-2.676	-2.481	-2.286	-2.092	-1.900	-1.708	-1.517	-1.328	-1.141	-0.956	-0.774	-0.596	-0.422	-0.253
0.850	-3.049	-2.852	-2.655	-2.459	-2.263	-2.068	-1.873	-1.679	-1.486	-1.294	-1.104	-0.915	-0.729	-0.545	-0.365	-0.189
0.900	-3.032	-2.834	-2.636	-2.438	-2.241	-2.044	-1.848	-1.651	-1.456	-1.262	-1.068	-0.875	-0.685	-0.496	-0.309	-0.126
0.925	-3.024	-2.825	-2.627	-2.428	-2.230	-2.033	-1.835	-1.638	-1.442	-1.246	-1.050	-0.856	-0.663	-0.471	-0.281	-0.094
0.950	-3.016	-2.817	-2.618	-2.419	-2.220	-2.022	-1.823	-1.625	-1.427	-1.230	-1.033	-0.837	-0.642	-0.447	-0.254	-0.063
0.960	-3.012	-2.813	-2.614	-2.415	-2.216	-2.017	-1.819	-1.620	-1.422	-1.224	-1.027	-0.830	-0.633	-0.438	-0.243	-0.050
0.970	-3.009	-2.810	-2.610	-2.411	-2.212	-2.013	-1.814	-1.615	-1.416	-1.218	-1.020	-0.822	-0.625	-0.428	-0.232	-0.038
0.975	-3.008	-2.808	-2.609	-2.409	-2.210	-2.011	-1.811	-1.612	-1.414	-1.215	-1.017	-0.818	-0.621	-0.424	-0.227	-0.031
0.980	-3.006	-2.807	-2.607	-2.407	-2.208	-2.009	-1.809	-1.610	-1.411	-1.212	-1.013	-0.815	-0.617	-0.419	-0.222	-0.025
0.990	-3.003	-2.803	-2.603	-2.404	-2.204	-2.004	-1.805	-1.605	-1.405	-1.206	-1.007	-0.807	-0.608	-0.409	-0.211	-0.013
0.995	-3.002	-2.802	-2.602	-2.402	-2.202	-2.002	-1.802	-1.602	-1.403	-1.203	-1.003	-0.804	-0.604	-0.405	-0.205	-0.006
Mean	-3.283	-3.098	-2.914	-2.732	-2.552	-2.373	-2.197	-2.024	-1.854	-1.688	-1.525	-1.367	-1.215	-1.069	-0.929	-0.798
Std Dev	0.267	0.278	0.291	0.306	0.321	0.338	0.356	0.376	0.398	0.421	0.446	0.473	0.503	0.534	0.568	0.603
Cov	-0.081	-0.090	-0.100	-0.112	-0.126	-0.142	-0.162	-0.186	-0.214	-0.249	-0.293	-0.346	-0.414	-0.500	-0.611	-0.756

Table 2. (continued)

$F(z)/k_z$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.005	-2.807	-2.759	-2.719	-2.685	-2.657	-2.635	-2.618	-2.605	-2.595	-2.588	-2.584	-2.581	-2.579	-2.577	-2.577	-2.576
0.010	-2.576	-2.525	-2.481	-2.444	-2.414	-2.390	-2.372	-2.358	-2.347	-2.340	-2.335	-2.332	-2.329	-2.328	-2.327	-2.327
0.020	-2.326	-2.271	-2.223	-2.183	-2.150	-2.124	-2.104	-2.088	-2.077	-2.069	-2.063	-2.060	-2.057	-2.056	-2.055	-2.054
0.025	-2.241	-2.184	-2.135	-2.094	-2.060	-2.033	-2.012	-1.996	-1.984	-1.976	-1.970	-1.966	-1.963	-1.962	-1.961	-1.961
0.030	-2.170	-2.111	-2.061	-2.018	-1.984	-1.956	-1.934	-1.918	-1.906	-1.897	-1.891	-1.887	-1.884	-1.884	-1.882	-1.881
0.040	-2.054	-1.992	-1.940	-1.895	-1.859	-1.830	-1.807	-1.789	-1.777	-1.768	-1.761	-1.757	-1.755	-1.753	-1.752	-1.751
0.050	-1.960	-1.896	-1.842	-1.795	-1.758	-1.727	-1.703	-1.685	-1.672	-1.663	-1.656	-1.652	-1.649	-1.647	-1.646	-1.646
0.075	-1.780	-1.712	-1.653	-1.603	-1.562	-1.529	-1.503	-1.484	-1.469	-1.459	-1.452	-1.447	-1.444	-1.442	-1.441	-1.440
0.100	-1.645	-1.572	-1.510	-1.457	-1.413	-1.378	-1.350	-1.329	-1.313	-1.302	-1.295	-1.290	-1.286	-1.284	-1.283	-1.282
0.150	-1.440	-1.360	-1.291	-1.233	-1.184	-1.145	-1.114	-1.090	-1.072	-1.060	-1.051	-1.045	-1.042	-1.039	-1.038	-1.037
0.200	-1.282	-1.196	-1.121	-1.057	-1.004	-0.961	-0.927	-0.901	-0.881	-0.868	-0.858	-0.852	-0.847	-0.845	-0.843	-0.843
0.250	-1.150	-1.059	-0.979	-0.910	-0.852	-0.805	-0.768	-0.739	-0.718	-0.703	-0.692	-0.685	-0.681	-0.678	-0.676	-0.676
0.300	-1.036	-0.939	-0.854	-0.780	-0.718	-0.667	-0.627	-0.595	-0.572	-0.556	-0.544	-0.536	-0.531	-0.528	-0.527	-0.526
0.400	-0.842	-0.733	-0.637	-0.553	-0.481	-0.422	-0.375	-0.338	-0.311	-0.291	-0.277	-0.268	-0.262	-0.258	-0.256	-0.255
0.500	-0.674	-0.554	-0.446	-0.351	-0.269	-0.200	-0.145	-0.101	-0.069	-0.045	-0.029	-0.017	-0.010	-0.006	-0.003	-0.002
0.600	-0.524	-0.392	-0.271	-0.163	-0.068	0.012	0.078	0.130	0.169	0.198	0.218	0.232	0.241	0.246	0.249	0.251
0.700	-0.385	-0.239	-0.103	0.020	0.130	0.225	0.304	0.368	0.417	0.453	0.479	0.497	0.508	0.515	0.519	0.522
0.750	-0.319	-0.165	-0.021	0.111	0.230	0.335	0.423	0.494	0.550	0.592	0.622	0.642	0.655	0.664	0.668	0.671
0.800	-0.253	-0.092	0.061	0.203	0.333	0.448	0.547	0.629	0.694	0.743	0.778	0.803	0.818	0.828	0.834	0.838
0.850	-0.189	-0.019	0.144	0.297	0.440	0.568	0.681	0.777	0.854	0.913	0.957	0.987	1.007	1.020	1.027	1.032
0.900	-0.126	0.053	0.227	0.394	0.551	0.697	0.829	0.944	1.039	1.115	1.173	1.213	1.241	1.258	1.269	1.275
0.925	-0.094	0.090	0.270	0.444	0.610	0.766	0.910	1.038	1.147	1.236	1.304	1.354	1.388	1.410	1.423	1.431
0.950	-0.063	0.126	0.312	0.494	0.671	0.839	0.997	1.142	1.270	1.378	1.464	1.528	1.574	1.603	1.622	1.633
0.960	-0.050	0.141	0.330	0.515	0.695	0.869	1.034	1.187	1.325	1.443	1.540	1.613	1.666	1.701	1.723	1.736
0.970	-0.038	0.156	0.347	0.536	0.721	0.901	1.073	1.235	1.384	1.515	1.625	1.712	1.775	1.818	1.846	1.862
0.975	-0.031	0.163	0.356	0.546	0.734	0.917	1.093	1.261	1.416	1.555	1.673	1.768	1.839	1.888	1.919	1.938
0.980	-0.025	0.170	0.365	0.557	0.747	0.933	1.113	1.286	1.449	1.596	1.725	1.830	1.911	1.967	2.005	2.027
0.990	-0.013	0.185	0.382	0.578	0.773	0.966	1.156	1.341	1.520	1.689	1.845	1.982	2.094	2.180	2.240	2.279
0.995	-0.006	0.193	0.391	0.589	0.786	0.983	1.178	1.370	1.559	1.742	1.917	2.078	2.222	2.340	2.430	2.492
Mean	-0.798	-0.675	-0.562	-0.459	-0.368	-0.288	-0.219	-0.163	-0.117	-0.082	-0.055	-0.036	-0.023	-0.014	-0.008	-0.004
Std Dev	0.603	0.640	0.678	0.717	0.756	0.794	0.830	0.863	0.894	0.920	0.942	0.959	0.972	0.982	0.989	0.993
Cov	-0.756	-0.948	-1.206	-1.561	-2.056	-2.759	-3.781	-5.301	-7.614	-11.23	-17.04	-26.66	-43.06	-71.96	-124.6	-223.8

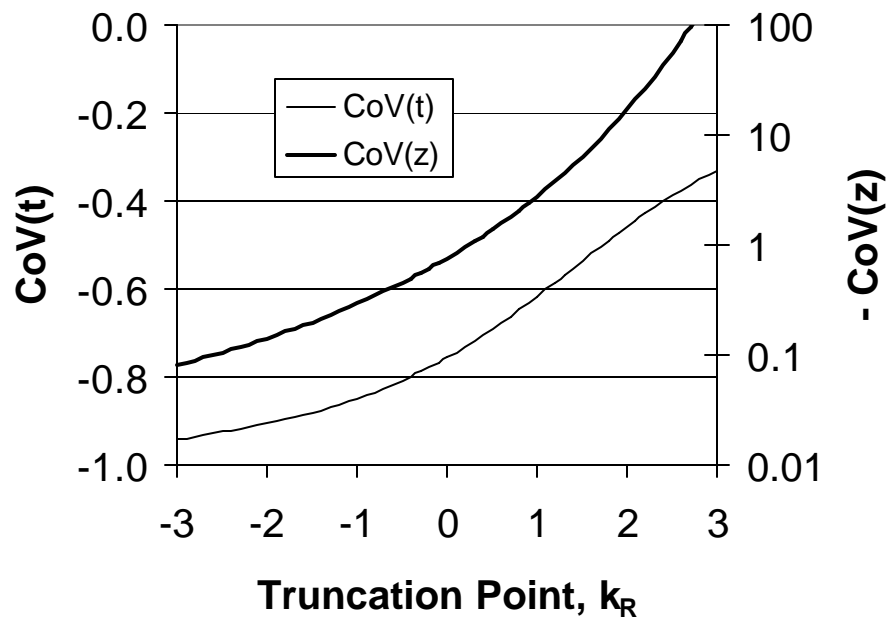


Figure 4. Coefficients of Variation in Terms of the Standardized Variate  $t$  and the Standard Normal Variate  $z$  as a Function of the Point of Truncation  $k_R$

### Doubly-Truncated Normal Distribution

**Definition.** For the case of a doubly-truncated normal distribution - i.e., one in which values of the normal variable below some point  $x_L$  and above some point  $x_R$  cannot be observed - the points of truncation  $x_L$  and  $x_R$  can be defined in terms of the standard normal variable  $z$  as shown in Equations 3 and 10 and denoted  $k_L$  and  $k_R$ , respectively.

Thus, the doubly-truncated normal distribution can be represented in terms of the standard normal distribution: (See Figure 5.)

$$f_{DTN}(z) = \begin{cases} 0, & -\infty \leq z \leq k_L \\ \frac{f(z)}{\int_{k_L}^{k_R} f(z) dz}, & k_L \leq z \leq k_R \\ 0, & k_R \leq z \leq \infty \end{cases}, \quad (16)$$

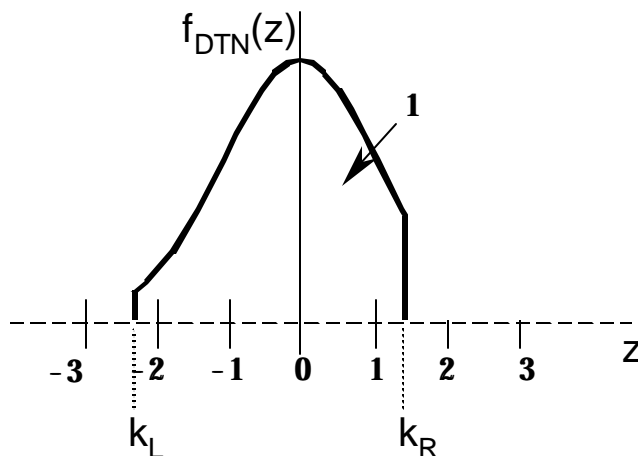


Figure 5. Doubly-Truncated Normal Distribution (in Terms of the Standard Normal Distribution)

From Equation 16, one may then find the cumulative distribution function of the doubly-truncated normal distribution, as shown below:

$$F_{\text{DTN}}(z) = \begin{cases} 0, & z \leq k_L \\ \frac{F(z) - F(k_L)}{F(k_R) - F(k_L)} & k_L \leq z \leq k_R \\ 1, & k_R \leq z \end{cases} \quad (17)$$

**Characteristic Parameters.** The mean ( $\mu_z$ ) and standard deviation ( $\sigma_z$ ) of the doubly-truncated normal distribution - as defined in Equation 16 - for given points of truncation  $k_L$  and  $k_R$  can be found in [Joh70] and elsewhere.

$$\mu_z = \frac{f(k_L) - f(k_R)}{F(k_R) - F(k_L)} \quad (18)$$

$$\sigma_z = \sqrt{E(z^2) - \mu_z^2} \quad (19)$$

where

$$E(z^2) = \frac{1}{F(k_R) - F(k_L)} \left[ -k_R f(k_R) + k_L f(k_L) + \{F(k_R) - F(k_L)\} \right]$$

From Equations 18 and 19, it is clear that the mean and standard deviation of the doubly-truncated normal distribution are solely dependent upon the points of truncation  $k_L$  and  $k_R$ . As before, the coefficient of variation  $c$  - which, again, is determined solely by the particular values of  $k_L$  and  $k_R$  - can be defined as

$$c = \frac{\sigma_z}{\mu_z} \quad (20)$$

**Reformulated Tables.** Using Equations 17 through 20, a prior table has been reformulated in terms of the standard normal variate ( $z$ ). This reformulated table is presented as Table 3 and provides the standard normal variate ( $z$ ) associated with a given value of the cumulative distribution function  $F(z)$ . Table 3 is divided into six segments - for  $F(z) = 0.01, 0.025, 0.05, 0.95, 0.975,$  and  $0.99$ .

As was noted elsewhere, the coefficient of variation  $c$  is not uniquely determined by the points of truncation  $k_L$  and  $k_R$  for the doubly-truncated normal distribution. This is illustrated in Figure 6, which presents a contour plot of the magnitude of the coefficient of variation over the range of allowable truncation points from  $-3$  to  $3$ .

In reviewing Figure 6, several points are worth noting. First, the figure possesses symmetry, as would be expected. Second, the "sign" of the coefficient of variation is determined by the relative degree of truncation of the lower and upper tails - which, in turn, determines the sign of the mean. Finally, the coefficient

of variation  $c$  approaches  $\pm\infty$  for cases of symmetric truncation and becomes undefined.



Table 3. (continued)

$k_t/k_s$		$F(z) = 0.025$															
		-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
-3.0	-2.993	-2.982	-2.963	-2.936	-2.897	-2.846	-2.784	-2.713	-2.636	-2.557	-2.478	-2.401	-2.329	-2.262	-2.202	-2.170	
-2.8	-2.793	-2.783	-2.766	-2.741	-2.708	-2.664	-2.611	-2.551	-2.486	-2.419	-2.352	-2.287	-2.226	-2.168	-2.119	-2.108	
-2.6	-2.583	-2.568	-2.546	-2.517	-2.480	-2.436	-2.386	-2.332	-2.277	-2.221	-2.174	-2.129	-2.086	-2.044	-2.019	-2.004	
-2.4	-2.384	-2.370	-2.351	-2.326	-2.295	-2.258	-2.218	-2.174	-2.134	-2.108	-2.078	-2.045	-2.011	-1.976	-1.942	-1.922	
-2.2	-2.194	-2.185	-2.172	-2.155	-2.134	-2.108	-2.078	-2.045	-2.011	-1.976	-1.942	-1.908	-1.874	-1.840	-1.806	-1.772	
-2.0	-1.985	-1.974	-1.959	-1.941	-1.919	-1.895	-1.869	-1.843	-1.816	-1.789	-1.762	-1.729	-1.710	-1.690	-1.669	-1.648	
-1.8	-1.794	-1.785	-1.774	-1.766	-1.759	-1.754	-1.750	-1.747	-1.744	-1.741	-1.738	-1.735	-1.732	-1.729	-1.726	-1.723	
-1.6	-1.594	-1.587	-1.582	-1.577	-1.572	-1.567	-1.562	-1.557	-1.552	-1.547	-1.542	-1.537	-1.532	-1.527	-1.522	-1.517	
-1.4	-1.394	-1.387	-1.382	-1.377	-1.372	-1.367	-1.362	-1.357	-1.352	-1.347	-1.342	-1.337	-1.332	-1.327	-1.322	-1.317	
-1.2	-1.194	-1.188	-1.183	-1.178	-1.173	-1.168	-1.163	-1.158	-1.153	-1.148	-1.143	-1.138	-1.133	-1.128	-1.123	-1.118	
-1.0	-0.994	-0.988	-0.983	-0.978	-0.973	-0.968	-0.963	-0.958	-0.953	-0.948	-0.943	-0.938	-0.933	-0.928	-0.923	-0.918	
-0.8	-0.794	-0.788	-0.783	-0.778	-0.773	-0.768	-0.763	-0.758	-0.753	-0.748	-0.743	-0.738	-0.733	-0.728	-0.723	-0.718	
-0.6	-0.594	-0.588	-0.583	-0.578	-0.573	-0.568	-0.563	-0.558	-0.553	-0.548	-0.543	-0.538	-0.533	-0.528	-0.523	-0.518	
-0.4	-0.394	-0.388	-0.383	-0.378	-0.373	-0.368	-0.363	-0.358	-0.353	-0.348	-0.343	-0.338	-0.333	-0.328	-0.323	-0.318	
-0.2	-0.194	-0.188	-0.183	-0.178	-0.173	-0.168	-0.163	-0.158	-0.153	-0.148	-0.143	-0.138	-0.133	-0.128	-0.123	-0.118	
$k_t/k_s$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
-3.0	-2.202	-2.149	-2.104	-2.065	-2.033	-2.007	-1.987	-1.972	-1.961	-1.953	-1.947	-1.944	-1.941	-1.940	-1.939	-1.938	
-2.8	-2.170	-2.121	-2.078	-2.041	-2.010	-1.986	-1.967	-1.952	-1.941	-1.933	-1.928	-1.925	-1.922	-1.921	-1.920	-1.920	
-2.6	-2.119	-2.074	-2.035	-2.001	-1.973	-1.950	-1.932	-1.918	-1.908	-1.901	-1.896	-1.893	-1.891	-1.889	-1.888	-1.888	
-2.4	-2.044	-2.005	-1.971	-1.941	-1.916	-1.895	-1.879	-1.867	-1.857	-1.851	-1.846	-1.843	-1.841	-1.840	-1.839	-1.839	
-2.2	-1.942	-1.910	-1.882	-1.856	-1.835	-1.817	-1.803	-1.792	-1.784	-1.779	-1.775	-1.772	-1.770	-1.769	-1.768	-1.768	
-2.0	-1.816	-1.791	-1.768	-1.747	-1.729	-1.715	-1.703	-1.694	-1.687	-1.682	-1.679	-1.676	-1.675	-1.674	-1.673	-1.673	
-1.8	-1.669	-1.650	-1.631	-1.615	-1.601	-1.589	-1.579	-1.572	-1.566	-1.562	-1.559	-1.557	-1.556	-1.555	-1.555	-1.555	
-1.6	-1.507	-1.492	-1.477	-1.464	-1.453	-1.443	-1.436	-1.430	-1.425	-1.422	-1.420	-1.418	-1.417	-1.417	-1.416	-1.416	
-1.4	-1.333	-1.321	-1.310	-1.300	-1.290	-1.283	-1.277	-1.272	-1.268	-1.266	-1.264	-1.262	-1.262	-1.261	-1.261	-1.261	
-1.2	-1.152	-1.142	-1.133	-1.125	-1.117	-1.111	-1.106	-1.102	-1.099	-1.097	-1.096	-1.095	-1.094	-1.094	-1.093	-1.093	
-1.0	-0.965	-0.957	-0.950	-0.943	-0.937	-0.932	-0.928	-0.924	-0.922	-0.920	-0.919	-0.918	-0.917	-0.917	-0.917	-0.917	
-0.8	-0.775	-0.769	-0.762	-0.756	-0.751	-0.747	-0.743	-0.740	-0.738	-0.737	-0.736	-0.735	-0.734	-0.734	-0.734	-0.734	
-0.6	-0.583	-0.577	-0.572	-0.566	-0.562	-0.558	-0.555	-0.552	-0.550	-0.549	-0.548	-0.547	-0.547	-0.547	-0.546	-0.546	
-0.4	-0.389	-0.384	-0.379	-0.374	-0.370	-0.366	-0.364	-0.361	-0.360	-0.358	-0.357	-0.357	-0.356	-0.356	-0.356	-0.356	
-0.2	-0.195	-0.190	-0.185	-0.181	-0.177	-0.173	-0.170	-0.168	-0.167	-0.165	-0.165	-0.164	-0.164	-0.163	-0.163	-0.163	
0.0	0.005	0.010	0.014	0.018	0.021	0.024	0.024	0.026	0.028	0.029	0.030	0.030	0.031	0.031	0.031	0.031	
0.2	0.205	0.209	0.213	0.217	0.220	0.222	0.223	0.223	0.223	0.225	0.226	0.226	0.226	0.227	0.227	0.227	
0.4	0.405	0.409	0.413	0.416	0.418	0.420	0.420	0.420	0.420	0.421	0.422	0.423	0.423	0.423	0.423	0.423	
0.6	0.605	0.609	0.612	0.615	0.617	0.618	0.618	0.618	0.618	0.619	0.619	0.620	0.620	0.620	0.621	0.621	
0.8	0.805	0.808	0.811	0.814	0.815	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816	0.816	
1.0	1.005	1.008	1.011	1.013	1.014	1.014	1.014	1.014	1.014	1.014	1.014	1.015	1.015	1.015	1.015	1.015	
1.2	1.204	1.208	1.210	1.211	1.211	1.211	1.211	1.211	1.211	1.211	1.212	1.213	1.214	1.214	1.215	1.215	
1.4	1.404	1.408	1.411	1.412	1.412	1.412	1.412	1.412	1.412	1.412	1.412	1.412	1.412	1.412	1.413	1.413	
1.6	1.604	1.608	1.611	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	1.612	
1.8	1.804	1.808	1.811	1.812	1.812	1.812	1.812	1.812	1.812	1.812	1.812	1.812	1.812	1.812	1.811	1.811	
2.0	2.004	2.008	2.011	2.012	2.012	2.012	2.012	2.012	2.012	2.012	2.012	2.012	2.012	2.012	2.010	2.010	
2.2	2.204	2.208	2.211	2.212	2.212	2.212	2.212	2.212	2.212	2.212	2.212	2.212	2.212	2.212	2.208	2.208	
2.4	2.404	2.408	2.411	2.412	2.412	2.412	2.412	2.412	2.412	2.412	2.412	2.412	2.412	2.412	2.406	2.406	
2.6	2.604	2.608	2.611	2.612	2.612	2.612	2.612	2.612	2.612	2.612	2.612	2.612	2.612	2.612	2.606	2.606	
2.8	2.804	2.808	2.811	2.812	2.812	2.812	2.812	2.812	2.812	2.812	2.812	2.812	2.812	2.812	2.806	2.806	
3.0	3.004	3.008	3.011	3.012	3.012	3.012	3.012	3.012	3.012	3.012	3.012	3.012	3.012	3.012	3.006	3.006	

Table 3. (continued)

$k_U/k_B$		$F(z) = 0.05$															
		-3.0	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2	0.0
-3.0	-1.938	-2.987	-2.965	-2.930	-2.882	-2.818	-2.739	-2.650	-2.554	-2.455	-2.357	-2.261	-2.170	-2.086	-2.008	-1.938	
-2.8	-1.920	-2.787	-2.766	-2.735	-2.691	-2.634	-2.564	-2.486	-2.406	-2.313	-2.226	-2.141	-2.061	-1.987	-1.920		
-2.6	-1.889	-2.587	-2.567	-2.537	-2.499	-2.449	-2.388	-2.320	-2.246	-2.170	-2.094	-2.021	-1.952	-1.889			
-2.4	-1.841	-2.387	-2.369	-2.342	-2.307	-2.263	-2.210	-2.151	-2.088	-2.024	-1.960	-1.898	-1.841				
-2.2	-1.772	-2.188	-2.170	-2.146	-2.114	-2.076	-2.031	-1.981	-1.928	-1.874	-1.822	-1.772					
-2.0	-1.699	-1.988	-1.971	-1.949	-1.921	-1.888	-1.849	-1.808	-1.766	-1.722	-1.679	-1.632	-1.597	-1.562			
-1.8	-1.616	-1.788	-1.772	-1.752	-1.727	-1.698	-1.666	-1.632	-1.598	-1.568	-1.533	-1.508	-1.481	-1.453	-1.425		
-1.6	-1.524	-1.508	-1.492	-1.476	-1.460	-1.444	-1.428	-1.412	-1.396	-1.380	-1.364	-1.348	-1.332	-1.316	-1.294	-1.272	
-1.4	-1.432	-1.416	-1.400	-1.384	-1.368	-1.352	-1.336	-1.320	-1.304	-1.288	-1.272	-1.256	-1.240	-1.224	-1.208	-1.192	
-1.2	-1.340	-1.324	-1.308	-1.292	-1.276	-1.260	-1.244	-1.228	-1.212	-1.196	-1.180	-1.164	-1.148	-1.132	-1.116	-1.100	
-1.0	-1.248	-1.232	-1.216	-1.200	-1.184	-1.168	-1.152	-1.136	-1.120	-1.104	-1.088	-1.072	-1.056	-1.040	-1.024	-1.008	
-0.8	-1.156	-1.140	-1.124	-1.108	-1.092	-1.076	-1.060	-1.044	-1.028	-1.012	-0.996	-0.980	-0.964	-0.948	-0.932		
-0.6	-1.064	-1.048	-1.032	-1.016	-1.000	-0.984	-0.968	-0.952	-0.936	-0.920	-0.904	-0.888	-0.872	-0.856	-0.840	-0.824	
-0.4	-0.972	-0.956	-0.940	-0.924	-0.908	-0.892	-0.876	-0.860	-0.844	-0.828	-0.812	-0.796	-0.780	-0.764	-0.748	-0.732	
-0.2	-0.880	-0.864	-0.848	-0.832	-0.816	-0.800	-0.784	-0.768	-0.752	-0.736	-0.720	-0.704	-0.688	-0.672	-0.656	-0.640	
0.0	-0.788	-0.772	-0.756	-0.740	-0.724	-0.708	-0.692	-0.676	-0.660	-0.644	-0.628	-0.612	-0.596	-0.580	-0.564	-0.548	
0.2	-0.696	-0.680	-0.664	-0.648	-0.632	-0.616	-0.600	-0.584	-0.568	-0.552	-0.536	-0.520	-0.504	-0.488	-0.472	-0.456	
0.4	-0.604	-0.588	-0.572	-0.556	-0.540	-0.524	-0.508	-0.492	-0.476	-0.460	-0.444	-0.428	-0.412	-0.396	-0.380	-0.364	
0.6	-0.512	-0.496	-0.480	-0.464	-0.448	-0.432	-0.416	-0.400	-0.384	-0.368	-0.352	-0.336	-0.320	-0.304	-0.288	-0.272	
0.8	-0.420	-0.404	-0.388	-0.372	-0.356	-0.340	-0.324	-0.308	-0.292	-0.276	-0.260	-0.244	-0.228	-0.212	-0.196	-0.180	
1.0	-0.328	-0.312	-0.296	-0.280	-0.264	-0.248	-0.232	-0.216	-0.200	-0.184	-0.168	-0.152	-0.136	-0.120	-0.104	-0.088	
1.2	-0.236	-0.220	-0.204	-0.188	-0.172	-0.156	-0.140	-0.124	-0.108	-0.092	-0.076	-0.060	-0.044	-0.028	-0.012	0.004	
1.4	-0.144	-0.128	-0.112	-0.096	-0.080	-0.064	-0.048	-0.032	-0.016	0.000	0.016	0.032	0.048	0.064	0.080	0.096	
1.6	-0.052	-0.036	-0.020	-0.004	0.012	0.028	0.044	0.060	0.076	0.092	0.108	0.124	0.140	0.156	0.172	0.188	
1.8	0.040	0.024	0.008	0.000	0.016	0.032	0.048	0.064	0.080	0.096	0.112	0.128	0.144	0.160	0.176	0.192	
2.0	0.128	0.112	0.096	0.080	0.064	0.048	0.032	0.016	0.000	0.016	0.032	0.048	0.064	0.080	0.096	0.112	
2.2	0.216	0.200	0.184	0.168	0.152	0.136	0.120	0.104	0.088	0.072	0.056	0.040	0.024	0.008	0.000	0.016	
2.4	0.304	0.288	0.272	0.256	0.240	0.224	0.208	0.192	0.176	0.160	0.144	0.128	0.112	0.096	0.080	0.064	
2.6	0.392	0.376	0.360	0.344	0.328	0.312	0.296	0.280	0.264	0.248	0.232	0.216	0.200	0.184	0.168	0.152	
2.8	0.480	0.464	0.448	0.432	0.416	0.400	0.384	0.368	0.352	0.336	0.320	0.304	0.288	0.272	0.256	0.240	
3.0	0.568	0.552	0.536	0.520	0.504	0.488	0.472	0.456	0.440	0.424	0.408	0.392	0.376	0.360	0.344	0.328	







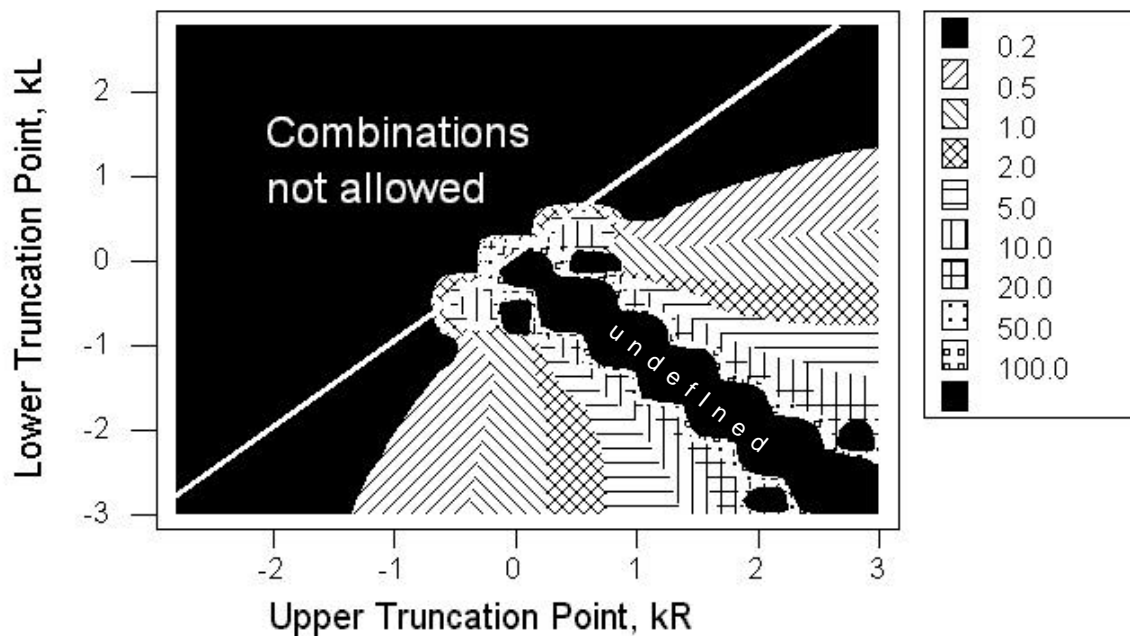


Figure 6. Contour Plot of the Magnitude of the Coefficient of Variation of the Doubly-Truncated Normal Distribution (in Terms of the Standard Normal Variate)

Use of the tables presented in this paper is analogous to that described in the author's other working papers for the corresponding tables in terms of standardized  $t$  values. These tables have been presented as a matter of convenience for researchers/workers who might prefer use of the standard normal variate  $z$  in their calculations.

### Combinations of Distributions

While this paper has focused on the case in which a single, otherwise-normally distributed pattern of variation is singly- or doubly-truncated, one can certainly utilize the results and tables presented in this paper to analyze combinations of truncated normal distributions. One example of a relatively general combination is presented in Figure 7.

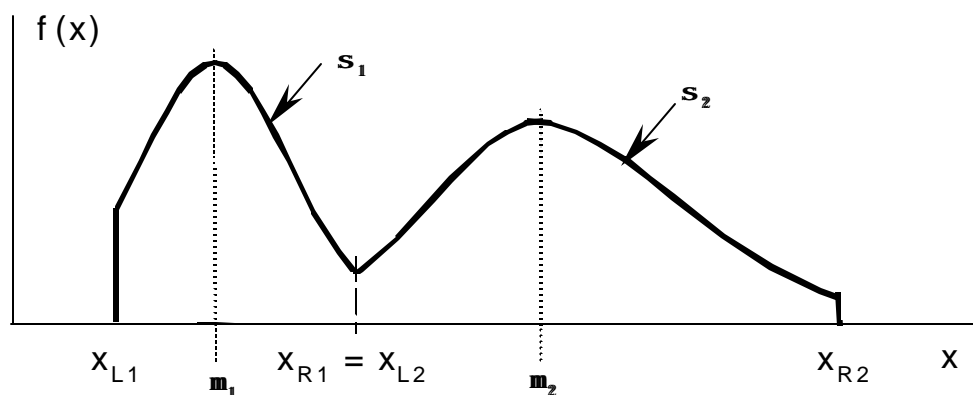


Figure 7. Combinations of Truncated Normal Distributions

The approach that one could utilize in analyzing a combination of truncated normal distributions - such as presented in Figure 7 - would be to utilize the results of this paper separately on the two constituent distributions. These individual results could, then, be combined after

"normalizing" for the relatively areas (i.e., probabilities) of the constituent distributions.

## BIBLIOGRAPHY

- [Abr72] Abramowitz, M., and Stegun, I., Handbook of Mathematical Functions, New York: Dover Publications, 1972.
- [Ait58] Aitchison, J., and Silvey, S., "Maximum-Likelihood Estimation of Parameters Subject to Restraints," Annals of Mathematical Statistics, Vol. 29, pp. 813-828, 1958.
- [Bar99] Barr, D., and Sherrill, E., "Mean and Variance of Truncated Normal Distributions," The American Statistician, Vol. 53, pp. 357-361, 1999.
- [Bey87] Beyer, W. (ed.), CRC Standard Mathematical Tables, Boca Raton: CRC Press, 28<sup>th</sup> Edition, 1987.
- [Cas94] Castillo, J., "The Singly Truncated Normal Distribution: A Non-Steep Exponential Family," Annals of the Institute of Statistical Mathematics, Vol. 46, pp. 57-66, 1994.
- [Chi81] Chiu, W., and Leung, M., "A Graphical Method for Estimating the Parameters of a Truncated Normal Distribution," Journal of Quality Technology, Vol. 13, pp. 42-45, 1981.
- [Coh49] Cohen, A., "On Estimating the Mean and Standard Deviation of Truncated Normal Distributions," Journal of the American Statistical Association, Vol. 44, pp. 518-525, 1949.
- [Coh50a] Cohen, A., "Estimating Parameters of Pearson Type III Populations From Truncated Samples," Journal of the American Statistical Association, Vol. 45, pp. 411-423, 1950.
- [Coh50b] Cohen, A., "Estimating the Mean and Variance of Normal Populations From Singly Truncated and Doubly Truncated Samples," Annals of

Mathematical Statistics, Vol. 21, pp. 557-569, 1950.

- [Coh50c] Cohen, A., "On Estimating the Mean and Variance of Singly Truncated Normal Frequency Distributions From the First Three Sample Moments," Annals of the Institute of Statistical Mathematics (Tokyo), Vol. 3, pp. 37-44, 1950.
- [Coh52] Cohen, A., "Estimation of Parameters in Truncated Pearson Frequency Distributions," Annals of Mathematical Statistics, Vol. 22, pp. 256-265, 1952.
- [Coh53] Cohen, A., and Woodward, J., "Tables of Pearson-Lee-Fisher Functions of Singly Truncated Normal Distributions," Biometrics, Vol. 9, pp. 489-497, 1953.
- [Coh55a] Cohen, A., "Censored Samples from Truncated Normal Distributions," Biometrika, Vol. 42, pp. 516-519, 1955.
- [Coh55b] Cohen, A., "Restriction and Selection in Samples From Bivariate Normal Distributions," Journal of the American Statistical Association, Vol. 50, pp. 884-893, 1955.
- [Coh57a] Cohen, A., "On the Solution of Estimating Equations for Truncated and Censored Samples From Normal Populations," Biometrika, Vol. 44, pp. 225-236, 1957.
- [Coh57b] Cohen, A., "Restriction and Selection in Multinormal Distributions," Annals of Mathematical Statistics, Vol. 28, pp. 731-741, 1957.
- [Coh59] Cohen, A., "Simplified Estimators for the Normal Distribution When Samples Are Singly Censored or Truncated," Technometrics, Vol. 1, pp. 217-237, 1959.
- [Coh61a] Cohen, A., "Estimating the Poisson Parameters From Samples That Are Truncated on the

- Right," Technometrics, Vol. 3, pp. 433-438, 1961.
- [Coh61b] Cohen, A., "Tables for Maximum Likelihood Estimates: Singly Truncated and Singly Censored Samples," Technometrics, Vol. 3, pp. 535-541, 1961.
- [Coh91] Cohen, A., Truncated and Censored Samples, New York: Marcel Dekker, 1991.
- [Cra79] Crain, B., "Estimating the Parameters of a Truncated Normal Distribution," Applied Mathematics and Computation, Vol. 5, pp. 149-156, 1979.
- [Fis30] Fisher, R., "The Moments of the Distribution of Normal Samples of Measures of Departure From Normality," Proceedings of the Royal Society of London, Vol. 130, pp. 16-28, 1930.
- [Fis31] Fisher, R., "The Truncated Normal Distribution," British Association for the Advancement of Science, Mathematical Tables, Vol. 5, pp. xxxiii-xxxiv, 1931.
- [Gup52] Gupta, A., "Estimation of the Mean and Standard Deviation of a Normal Population From a Censored Sample," Biometrika, Vol. 39, pp. 260-273, 1952.
- [Hal49] Hald, A., "Maximum Likelihood Estimation of the Parameters of a Normal Distribution Which is Truncated at a Known Point," Skandinavisk Aktuarietidskrift, Vol. 32, pp. 119-134, 1949.
- [Hal52] Hald, A., Statistical Theory with Engineering Applications, New York: John Wiley and Sons, 1952.
- [Hal53] Halperin, M., "Estimation in the Truncated Normal Distribution," Journal of the American Statistical Association, Vol. 47, pp. 457-465, 1953.

- [Han80] Hansen, J., and Zeger, S., "The Asymptotic Variance of the Estimated Proportion Truncated From a Normal Population," Technometrics, Vol. 22, pp. 271-274, 1980.
- [Har66] Harter, H., and Moore, A., "Iterative Maximum Likelihood Estimation of the Parameters of Normal Populations From Singly and Doubly Censored Samples," Biometrika, Vol. 53, pp. 205-213, 1966.
- [Heg89] Hegde, L., and Dahiya, R., "Estimation of Parameters in a Truncated Normal Distribution," Communications in Statistics - Theory and Methods, Vol. 18, pp. 4177-4195, 1989.
- [Hin90] Hines, W., and Montgomery, D., Probability and Statistics in Engineering and Management Science, New York: John Wiley and Sons, 1990.
- [Joh70] Johnson, N., and Kotz, S., Distributions in Statistics: Continuous Univariate Distributions 1, New York: Houghton Mifflin, 1970.
- [Kec91] Kececioglu, D., Reliability Engineering Handbook, Vol. 1, Englewood Cliffs: Prentice Hall, 1991.
- [Kha79] Khatri, C., and Ratani, R., "On Estimation of the Mean Parameter of a Truncated Normal Distribution With Known Coefficient of Variation," Communications in Statistics - Theory and Methods, Vol. 8, pp. 237-244, 1979.
- [Law91] Law, A., and Kelton, W., Simulation Modeling and Analysis, New York: McGraw-Hill, 1991.
- [Mit87] Mittal, M., and Dahiya, R., "Estimating the Parameters of a Doubly Truncated Normal Distribution," Communications in Statistics - Simulation, Vol. 16, pp. 141-159, 1987.

- [Nor80] Norgaard, R., and Killeen, T., "Expected Utility and the Truncated Normal Distribution," Management Science, Vol. 26, pp. 901-909, 1980.
- [Pat82] Patel, J., and Read, C., Handbook of the Normal Distribution, New York: Marcel Dekker, 1982.
- [Pea55] Pearson, K., "The Normal Probability Function: Tables of Certain Area-Ordinate Ratios and of Their Reciprocals," Biometrika, Vol. 42, pp. 217-222, 1955.
- [Raj53] Raj, D., "Estimation of the Parameters of Type III Populations From Truncated Samples," Journal of the American Statistical Association, Vol. 48, pp. 336-349, 1953.
- [Rob64] Robson, D., and Whitlock, J., "Estimation of a Truncation Point," Biometrika, Vol. 51, pp. 33-39, 1964.
- [Sar56] Sarhan, A., and Greenberg, B., "Estimation of Location and Scale Parameters by Order Statistics From Single and Doubly Censored Samples," Annals of Mathematical Statistics, Vol. 27, pp. 427-451, 1956.
- [Sch86] Schneider, H., Truncated and Censored Samples From Normal Populations, New York: Marcel Dekker, 1986.
- [Sha66] Shah, S., and Jaiswal, M., "Estimation of Parameters of Doubly Truncated Normal Distribution From First Four Sample Moments," Annals of the Institute of Statistical Mathematics (Tokyo), Vol. 18, pp. 107-111, 1966.
- [Ste93] Stein, W., Pfaffenberger, R., and Mizzi, P., "A Stochastic Dominance Comparison of Truncated Normal Distributions," European Journal of Operational Research, Vol. 67, pp. 259-266, 1993.
- [Tho80] Thomopoulos, N., Applied Forecasting Methods, Englewood Cliffs: Prentice-Hall, 1980.